<u>Example</u> Find general solution of the given system of equations:

$$x' = \underbrace{\begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}}_{A} x \qquad \begin{cases} x_1' = x_1 - x_2 + 4x_3 \\ x_2' = 3x_1 + 2x_2 - x_3 \\ x_3' = 2x_1 + x_2 - x_3 \end{cases}$$

Solution

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -1 & 4 \\ 3 & 2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{vmatrix} =$$

char. pol. for $\begin{pmatrix} 2 & -1 \\ 1 & -1 \end{pmatrix}$

$$= (1-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ 2 & -1-\lambda \end{vmatrix} + 4\begin{vmatrix} 3 & 2-\lambda \\ 2 & 1 \end{vmatrix} =$$

$$= (1-\lambda)(\lambda^2 - \lambda - 1) - 3 - 3\lambda + 2 + 4(3 - 4 + 2\lambda)$$

$$= \lambda^2 \cancel{-\lambda} - 1 - \lambda^3 + \lambda^2 \cancel{+\lambda} - 3\lambda - 1 - 4 + 8\lambda$$

$$= -\lambda^3 + 2\lambda^2 + 5\lambda - 6 = 0$$

The candidates for roots are divisors of the free term:

$$\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$$

which are $\pm 1, \pm 2, \pm 3, \pm 6$

$\lambda = -2 \Rightarrow -8 - 8 + 10 + 6 = 0$

$$\begin{array}{r|l} & \lambda^2 - 4\lambda + 3 \\ \lambda + 2 & \lambda^3 - 2\lambda^2 - 5\lambda + 6 \\ & \lambda^3 + 2\lambda^2 \\ \hline & -4\lambda^2 - 5\lambda \\ & -4\lambda^2 - 8\lambda \\ \hline & 3\lambda + 6 \end{array}$$

$(\lambda + 2)(\lambda^2 - 4\lambda + 3) = 0$

$(\lambda + 2)(\lambda - 1)(\lambda - 3) = 0$

$\lambda_1 = 1, \ \lambda_2 = -2, \ \lambda_3 = 3$

Step 2 Find eigen vectors

(a) $\lambda_1 = 1$ $(A - I)V = 0 \Rightarrow \begin{pmatrix} 0 & -1 & 4 \\ 3 & 1 & -1 \\ 2 & 1 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0$

Use Gauss Elimination method (get matrix with elements under diagonal equals zero. Use elementary row operations on augmented matrix which corresponds to legitime operations on the equations in the system.

The augmented matrix is

$$\left(\begin{array}{ccc|c} 0 & -1 & 4 & 0 \\ 3 & 1 & -1 & 0 \\ 2 & 1 & -2 & 0 \end{array}\right)$$

$\downarrow R_1 \leftrightarrow R_2$

$$\left(\begin{array}{ccc|c} 3 & 1 & -1 & 0 \\ 0 & -1 & 4 & 0 \\ 2 & 1 & -2 & 0 \end{array}\right) \xrightarrow{R_3 \leftrightarrow 3R_3 - 2R_1} \left(\begin{array}{ccc|c} 3 & 1 & -1 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & 1 & -4 & 0 \end{array}\right) \xrightarrow{R_3 \leftarrow R_3 + R_2}$$

$$\rightarrow \left(\begin{array}{ccc|c} 3 & 1 & -1 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} 3v_1 + v_2 - v_3 = 0 \\ -v_2 + 4v_3 = 0 \end{array}$$

If $v_3 = 1 \Rightarrow v_2 = +4v_3 = +4$

$3v_1 = v_3 - v_2 = 1 - (+4) = -3 \Rightarrow v_1 = -1$

$V = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} \Rightarrow$ solution $e^t \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$

(b) $\lambda = -2$

$$A - \lambda I = A + 2I \Rightarrow \text{augm. matrix} \left(\begin{array}{ccc|c} 3 & -1 & 4 & 0 \\ 3 & 4 & -1 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right) \to$$

$$\xrightarrow{R_2 \leftrightarrow R_2 - R_1} \left(\begin{array}{ccc|c} 3 & -1 & 4 & 0 \\ 0 & 5 & -5 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right) \xrightarrow{R_3 \leftrightarrow 3R_3 - 2R_1} \left(\begin{array}{ccc|c} 3 & -1 & 4 & 0 \\ 0 & 5 & -5 & 0 \\ 0 & 5 & -5 & 0 \end{array}\right)$$

$$\xrightarrow{R_3 \leftrightarrow R_3 - R_2} \left(\begin{array}{ccc|c} 3 & -1 & 4 & 0 \\ 0 & 5 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 3 & -1 & 4 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right)$$

$$3V_1 - V_2 + 4V_3 = 0$$
$$V_2 - V_3 = 0$$

If $V_3 = 1 \Rightarrow V_2 = 1 \Rightarrow 3V_1 - 1 + 4 = 0$

$$3V_1 = 3$$
$$V_1 = 1$$

$$V = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \text{Solution } e^{-2t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(c) $\lambda = 3 \Rightarrow A - 3I = \left(\begin{array}{ccc|c} -2 & -1 & 4 & 0 \\ 3 & -1 & -1 & 0 \\ 2 & 1 & -4 & 0 \end{array}\right) \to$

$$\xrightarrow{R_3 \leftrightarrow R_3 + R_1} \left(\begin{array}{ccc|c} -2 & -1 & 4 & 0 \\ 3 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_2 \leftrightarrow 2R_2 + 3R_1} \begin{pmatrix} -2 & -1 & 4 \\ 0 & -5 & 10 \end{pmatrix}$$

$$-2V_1 - V_2 + 4V_3 = 0$$
$$-5V_2 + 10V_3 = 0$$

$V_3 = 1 \Rightarrow 5V_2 = 10 \Rightarrow V_2 = 2$

$\Rightarrow -2V_1 - 2 + 4 = 0$

$V_1 = V_1 = 1$

$v = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ solution $e^{3t}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

Step3 General solution

$$x(t) = C_1 e^{t}\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} + C_2 e^{-2t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + C_3 e^{3t}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$