MATH 308 Homework assignment 1 Solution

1) (a) V_e must satisfy the following equation

$$19.6 - \frac{V_e}{10} = 0 \Rightarrow \boxed{V_e = 196}$$

(b) $$v' = 19.6 - \frac{v}{10}$$

$$\frac{v'}{19.6 - \frac{v}{10}} = 1$$

Integrate w.r.t. t. $$\int \frac{v'dt}{19.6 - \frac{v}{10}} = \int dt \Rightarrow$$

$$\int \frac{dv\, dt}{19.6 - \frac{v}{10}} = t + C_1 \Rightarrow$$

$u = 19.6 - \frac{v}{10}$

$du = -\frac{dv}{10}$

$$-10\int \frac{du}{u} = t + C_1 \Rightarrow$$

$$-10 \ln|u| = t + C_1 \Rightarrow \ln\left|19.6 - \frac{v}{10}\right| = -\frac{t}{10} + C_2 \Rightarrow$$

$$\left|19.6 - \frac{v}{10}\right| = \underbrace{e^{C_2}}_{C_3 > 0} e^{-\frac{t}{10}} \Rightarrow 19.6 - \frac{v}{10} = C_3 e^{-t/10}, \quad C_3 \in \mathbb{R} \Rightarrow$$

$$\Rightarrow \frac{v}{10} = 196 - c_3 e^{-t/10} \Rightarrow$$ the general solution is

$$v(t) = 196 - Ce^{-t/10}$$

If $v(0) = 0$ then $0 = 196 - C \Rightarrow C = 196 \Rightarrow$

The solution of the initial value problem is

$$\boxed{v(t) = 196\left(1 - e^{-t/10}\right)}$$

$$\boxed{\lim_{t \to +\infty} v(t) = 196 = V_e}$$

(c) We have to solve the equation

$$196\left(1-e^{-t/10}\right) = \frac{1}{2}196 \Leftrightarrow$$

$$1-e^{-t/10} = \frac{1}{2} \Leftrightarrow e^{-t/10} = \frac{1}{2} \Rightarrow$$

$$-\frac{t}{10} = \ln\frac{1}{2} = -\ln 2 \Rightarrow \frac{t}{10} = \ln 2 \Rightarrow$$

$$\boxed{t = 10\ln 2}$$

(d) Let $s(t)$ be the distance the object has fallen in time t. $\Rightarrow$ $s'(t) = v(t) \Rightarrow$

$$s(10\ln 2) = \int_0^{10\ln 2} v(t)\,dt = \int_0^{10\ln 2} 196\left(1-e^{-t/10}\right)dt =$$

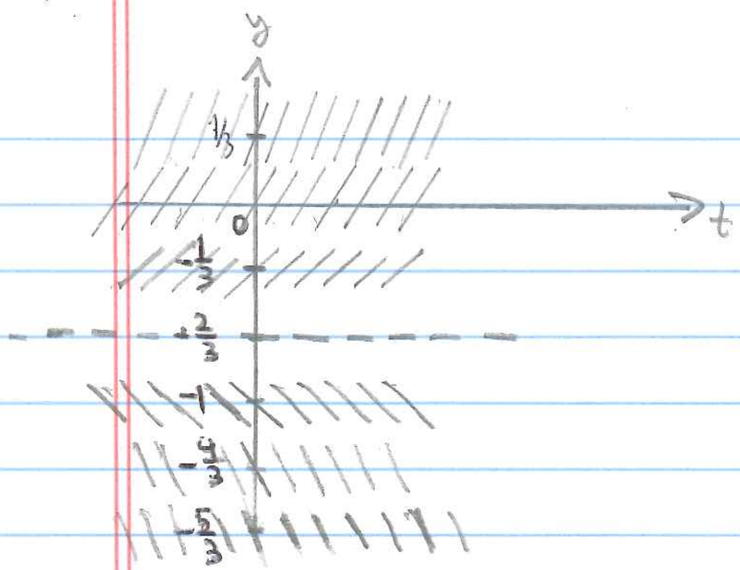
$$= 196\left(t + 10e^{-t/10}\right)\Big|_0^{10\ln 2} = 196\Big(10\ln 2 + \underbrace{10e^{-\ln 2}}_{10\cdot\frac{1}{2}} - 10\Big) = \boxed{196\left(10\ln 2 - 5\right)}$$

Problem 2

(a) Equillibrium point: $2+3y=0 \Rightarrow y=-\frac{2}{3} \Rightarrow$ equillibrium solution is $y(t) \equiv -\frac{2}{3}$

(b) $y' = 2+3y$

y	$-\frac{5}{3}$	$-\frac{4}{3}$	-1	$-\frac{2}{3}$	$-\frac{1}{3}$	0	$\frac{1}{3}$
slope	-3	-2	-1	0	1	2	3



(c) If $y_0 > -\frac{2}{3}$ $\quad \lim_{t \to +\infty} y(t) = +\infty$

If $y_0 < -\frac{2}{3}$ $\quad \lim_{t \to +\infty} y(t) = -\infty$

If $y \neq -\frac{2}{3}$ $\quad \lim_{t \to -\infty} y(t) = -\frac{2}{3}$

(d) $y' = 2+3y \iff \frac{y'}{2+3y} = 1 \iff \frac{dy}{2+3y} = dt \iff$

$$\underbrace{\int \frac{dy}{2+3y}}_{} = t + C_1$$

$$\frac{1}{3} \ln|2+3y| = t + C_1 \underset{\text{as in } 1(e)}{\iff} 2+3y = C_2 e^{3t} \Rightarrow$$

$$y(t) = -\frac{2}{3} + Ce^{3t}$$

If $y(0) = y_0$ then $y_0 = -\frac{2}{3} + C \Rightarrow C = y_0 + \frac{2}{3} \Rightarrow$

$$\boxed{y(t) = -\frac{2}{3} + \left(y_0 + \frac{2}{3}\right)e^{3t}}$$

From this explicit form

$$\lim_{t\to+\infty} y(t) = \begin{cases} +\infty & \text{if } y_0+\frac{2}{3}>0 \Leftrightarrow y_0>-\frac{2}{3} \\ -\infty & \text{if } y_0+\frac{2}{3}<0 \Leftrightarrow y_0<-\frac{2}{3} \end{cases}$$

$$\lim_{t\to-\infty} y(t) = -\frac{2}{3} \quad \text{as in item 1c)}$$

Problem 3

(a) $(1-x^2)^{1/2}y' + xy = 0$

$$y' = -\frac{xy}{(1-x^2)^{1/2}} \to \text{separable equation}$$

$$\frac{dy}{y} = -\frac{x\,dx}{(1-x^2)^{1/2}} \Leftrightarrow \int \frac{dy}{y} = -\underbrace{\int \frac{x\,dx}{(1-x^2)^{1/2}}} \Leftrightarrow$$

$$\ln|y| = -(1-x^2)^{1/2} + C_1$$

$$\Updownarrow$$

$$\boxed{y = Ce^{-(1-x^2)^{1/2}}}$$

$u = 1-x^2$, $du = -2x\,dx$: $\int \frac{x\,dx}{(1-x^2)^{1/2}} = -\frac{1}{2}\int u^{-1/2}\,du = -\frac{1}{2}\cdot(-2)\,u^{1/2} = u^{1/2} = (1-x^2)^{1/2}$

(b) $y' + \dfrac{x\cos x}{y\cos y} = 0 \Leftrightarrow y' = -\dfrac{x\cos x}{y\cos y} \Leftrightarrow$

Separable equation

$$\Leftrightarrow y\cos y\,dy = -x\cos x\,dx$$

$$\int y\cos y\,dy = -\int x\cos x\,dx + C$$

$$\Updownarrow$$

$$\boxed{y\sin y + \cos y = -x\sin x - \cos x + C}$$

$\int x\cos x\,dx \underset{\text{by part}}{=} x\sin x - \int \sin x\,dx = x\sin x + \cos x$

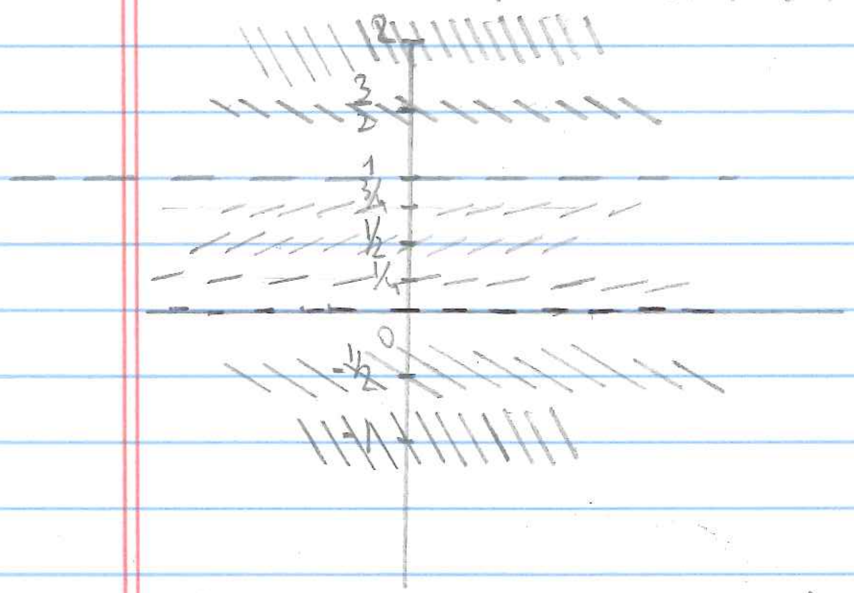
Problem 4

(a) Equillibrium point : $y - y^2 = 0 \Leftrightarrow$
$y(1-y) = 0 \Rightarrow y = 0$ or $y = 1$

(b) $y(1-y) > 0$ $\quad \overset{-}{}\underset{0}{|}\overset{+}{}\underset{1}{|}\overset{-}{}\rightarrow y$

Slope is positive if $0 < y < 1$
Slope is negative if $y < 0$ or $y > 1$

y	-1	$-\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{3}{2}$	2
slope	-2	$-\frac{3}{4}$	0	$\frac{3}{16}$	$\frac{1}{4}$	$\frac{3}{16}$	0	$-\frac{3}{4}$	-2



(c) i If $y(0) = \frac{1}{2}$ since $0 < y(0) < 1$ then by uniqueness $0 < y(t) < 1$ for any t (here $f(y) = y - y^2$ $\frac{\partial f}{\partial y} = 1 - 2y$ is continuous $\Rightarrow$ uniqueness holds) $\Rightarrow$ $y'(t)$ is positive for any t (the slope is positive on the strip $0 < y < 1$) $\Rightarrow$ $y(t)$ is monotonically increasing $\Rightarrow$ $\lim_{t \to +\infty} y(t)$ and $\lim_{t \to -\infty} y(t)$ exist and since it must be equal to an equillibrium point then

$$\lim_{t\to+\infty} y(t) = 1$$

$$\lim_{t\to-\infty} y(t) = 0$$

(c) (ii) First, The same arguments will work for all $0<y_0<1$

Second, If $y_0 \geq 1 \Rightarrow y(0) \geq 1 \Rightarrow$ by uniqueness $y(t) \geq 1$ for any $t \Rightarrow y'(t) < 0$ because the slope is negative in the strip $y>1 \Rightarrow y(t)$ is monotonically decreasing & $y(t) \geq 1 \Rightarrow \lim y(t)$ exists and since the limit must be equal to an equillibrium point $\lim_{t\to+\infty} y(t) = 1$ again

Third, If $y_0 < 0$ then by similar arguments $y(t)$ is monotonically decreasing $\Rightarrow y(t)$ does not go to 1 when $t \to +\infty$

Conclusion $y(t) \xrightarrow[t\to+\infty]{} 1 \Leftrightarrow \boxed{y_0 > 0}$

(c) (iii) As mentioned in the previous item $y(t)$ is monotonically decreasing. Also its slopes go to $-\infty \Rightarrow$

$\boxed{y(t) \text{ approach the value } -\infty \text{ as } t \text{ increases}}$

(d) bonus $y' = y - y^2 \Leftrightarrow \frac{dy}{y-y^2} = dt \Leftrightarrow$

$$\int \frac{dy}{y-y^2} = \int dt$$

$$\frac{1}{y-y^2} = \frac{1}{y(1-y)} = \frac{1}{y} + \frac{1}{1-y} \Rightarrow \int \frac{dy}{y-y^2} = \ln|y| - \ln|1-y| = t + C_1$$

$$\ln\left|\frac{y}{1-y}\right| = t + C_1 \Rightarrow$$

$$\frac{y}{1-y} = ce^t$$

Initial condition : $y(0) = -1 \Rightarrow$

$$\frac{-1}{2} = c \Rightarrow c = -\frac{1}{2} \Rightarrow$$

$$\frac{y}{1-y} = -\frac{1}{2}e^t \Rightarrow y = \frac{1}{2}e^t y - \frac{1}{2}e^t \Rightarrow$$

$$y(t) = \frac{\frac{1}{2}e^t}{\frac{1}{2}e^t - 1} = \boxed{\frac{e^t}{e^t-2}}$$

$e^t \neq 2 \Rightarrow t \neq \ln 2$. $\lim_{t \to \ln 2^-} y(t) = -\infty$

Since the initial time is $t_0 = 0$, then the solution is defined for $\boxed{t < \ln 2}$. Approaching to $\ln 2$ from the left $\boxed{\text{it exploses to } -\infty}$