MATH 308 Homework 6 Solution

1 1a) $\sinh bt = \frac{e^{bt}-e^{-bt}}{2} \Rightarrow$

$$\mathcal{L}\{\sinh bt\} = \int_0^\infty \sinh bt\, e^{-st}dt = \int_0^\infty \frac{e^{bt}-e^{-bt}}{2}e^{-st}dt =$$

$$= \frac{1}{2}\int_0^\infty \left(e^{(b-s)t} - e^{-(b+s)t}\right)dt = \frac{1}{2}\lim_{A\to\infty}\left(\frac{e^{(b-s)t}}{b-s} + \frac{e^{-(b+s)t}}{b+s}\right)\Bigg|_{t=0}^{A} = (*)$$

The finite limit exists for $\begin{cases} b-s<0 \\ b+s>0\end{cases} \Leftrightarrow \begin{cases} s>b \\ s>-b\end{cases} \Leftrightarrow s>|b|$

$$(*) = \frac{1}{2}\left(-\frac{1}{b-s} - \frac{1}{b+s}\right) = \frac{1}{2}\left(\frac{1}{s-b} - \frac{1}{s+b}\right) = \frac{1}{2}\,\frac{s+b-(s-b)}{(s-b)(s+b)} = \frac{2b}{2(s-b)(s+b)} = \boxed{\frac{b}{s^2-b^2}}, \ \boxed{|s|>b}$$

(b) $e^{at}\cosh bt = e^{at}\frac{e^{bt}+e^{-bt}}{2} = \frac{e^{(a+b)t}+e^{(a-b)t}}{2}$

$$\mathcal{L}\{e^{at}\cosh bt\} = \int_0^\infty e^{at}\cosh bt\, e^{-st}dt = \int_0^\infty \frac{e^{(a+b-s)t}+e^{(a-b-s)t}}{2} =$$

$$= \frac{1}{2}\lim_{A\to\infty}\left(\frac{e^{(a+b-s)t}}{a+b-s} + \frac{e^{(a-b-s)t}}{a-b-s}\right)\Bigg|_{t=0}^{A}$$

The finite limit exists for $\begin{cases} a+b-s<0 \\ a-b-s<0\end{cases} \Leftrightarrow \begin{cases} s-a>b \\ s-a>-b\end{cases} \Leftrightarrow s-a>|b|$

$$(*) = -\frac{1}{2}\left(\frac{1}{a+b-s} + \frac{1}{a-b-s}\right) = -\frac{1}{2}\,\frac{a-b-s+a+b-s}{(a-s+b)(a-s-b)} = -\frac{a-s}{(a-s)^2-b^2} = \frac{s-a}{(s-a)^2-b^2}$$

<u>Remark</u> Of course, both items can be solved almost immediately using the linearity, the fact that $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$, and, for the item b), the property of translation in s:

a) $\mathcal{L}\{\sinh bt\} = \mathcal{L}\left\{\frac{e^{bt}-e^{-bt}}{2}\right\} = \frac{1}{2}\left(\mathcal{L}\{e^{bt}\} - \mathcal{L}\{e^{-bt}\}\right) =$

$= \frac{1}{2}\left(\frac{1}{s-b} - \frac{1}{s+b}\right) = \frac{b}{s^2-b^2}$

b) $\mathcal{L}\{e^{at}\cosh bt\} = F(s-a)$ where $F(s) = \mathcal{L}\{\cosh bt\} =$

$= \frac{1}{2}\left(\mathcal{L}\{e^{bt}\} + \mathcal{L}\{e^{-bt}\}\right) = \frac{1}{2}\left(\frac{1}{s-b}+\frac{1}{s+b}\right) = \frac{s}{s^2-b^2} \Rightarrow \mathcal{L}\{e^{at}\cosh bt\} = \frac{s-a}{(s-a)^2-b^2}$

But we wanted you will use the definition of the Laplace transform here

2. a) Factor the denominator

$s^2+2s-8=0$

$D = 4 - 4\cdot(-8) = 4+32=36$

$s_1 = \frac{-2+6}{2} = 2$

$s_2 = \frac{-2-6}{2} = -4$

$\Rightarrow (s^2+2s-8) = (s-2)(s+4)$

(you can guess it of course but I prefer the use of quadratic formula)

$\Rightarrow$ The partial fraction decomposition is of the form

$$\frac{3s}{s^2+2s-8} = \frac{A}{s-2} + \frac{B}{s+4} \Rightarrow$$

$$3s = A(s+4) + b(s-2)$$

If $s=2$: $3\cdot 2 = A(2+4) \Rightarrow 6 = 6A \Rightarrow A=1$

If $s=-4$: $-3\cdot 4 = B(-4-2) \Rightarrow -12 = -6B \Rightarrow B=2$

$$\Downarrow$$

$$\frac{3s}{s^2+2s-8} = \frac{1}{s-2} + \frac{2}{s+4} \Rightarrow$$

$$\mathcal{L}^{-1}\left(\frac{3s}{s^2+2s-8}\right) = \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{s+4}\right\} = \boxed{e^{2t}+2e^{-4t}}$$

b) $F(s) = \frac{2s+5}{s^2+6s+25}$

Factor the denominator if possible

$s^2+6s+25=0$

$D = 36-100 < 0 \Rightarrow$ complex roots $\rightarrow$ the factorization is not possible over $\mathbb{R}$

Complete squares

$s^2+6s+25 = s^2+6s+9+16 = (s+3)^2+4^2$

We have to write F(s) in the following form

$$\frac{2s+5}{s^2-2s+5} = \frac{A(s+3)+B\cdot 4}{(s+3)^2+4^2} \Rightarrow$$

$$2s+5 = A(s+3)+4B = As+4B+3A \Rightarrow \text{equating coefficients}$$

$$A=2 \Rightarrow 4B = 5-3A = 5-6 = -1 \Rightarrow B = -\frac{1}{4}$$

$$\Rightarrow \frac{2s+5}{s^2-2s+5} = 2\,\frac{s+3}{(s+3)^2+4^2} - \frac{1}{4}\,\frac{4}{(s+3)^2+4^2} \Rightarrow$$

$$\mathcal{L}^{-1}\left(\frac{2s+5}{s^2-2s+5}\right) = 2\,\mathcal{L}^{-1}\left\{\frac{s+3}{(s+3)^2+4^2}\right\} - \frac{1}{4}\,\mathcal{L}^{-1}\left\{\frac{4}{(s+3)^2+4^2}\right\} =$$

$$= \boxed{2e^{-3t}\cos 4t - \frac{1}{4}e^{-3t}\sin 4t}$$

(here we used the fact that $\mathcal{L}^{-1}\left\{\frac{s}{s^2+4^2}\right\} = \cos 4t$, $\mathcal{L}^{-1}\left\{\frac{4}{s^2+4^2}\right\} = \sin 4t$ + Property of translation in s.

3) a) Apply the Laplace transform to both parts of the equation

$$\mathcal{L}\{y'\} = sY(s) - \underbrace{y(0)}_{0} = sY(s)$$

$$\mathcal{L}\{y''\} = s^2Y(s) - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{-1} = s^2Y(s)+1$$

$$\mathcal{L}\{y''-3y'+2y\} = s^2Y(s)+1-3sY(s)+2Y(s) = (s^2-3s+2)Y(s)+1 =$$

$$= \mathcal{L}\{\cos t\} = \frac{s}{s^2+1} \Rightarrow$$

$$(s^2-3s+2)Y(s)+1 = \frac{s}{s^2+1} \Rightarrow$$

$$\Rightarrow \underbrace{(s^2-3s+2)}_{(s-1)(s-2)}Y(s) = \frac{s}{s^2+1} - 1 = \frac{s-s^2-1}{s^2+1} \Rightarrow$$

$$\boxed{Y(s) = \frac{-s^2+s-1}{(s-1)(s-2)(s^2+1)}}$$

3 b) $y''+y'-y=t^3,\ y(0)=1,\ y'(0)=0$

$\mathcal{L}\{y'\} = sY(s) - \underbrace{y(0)}_{} = sY(s)-1$

$\mathcal{L}\{y''\} = s^2Y(s) - s\underbrace{y(0)}_{1} - \underbrace{y'(0)}_{0} = s^2Y(s)-s$

$\Downarrow$

$\mathcal{L}\{y''+y'-y\} = (s^2+s-1)Y(s) - s - 1 = \mathcal{L}\{t^3\} = \dfrac{6}{s^4}$

$\downarrow$ the table or derivation of Laplace transform

$\left(\mathcal{L}\{t^3\} = \mathcal{L}\{t^3\cdot 1\} = (-1)^3\left(\frac{1}{s}\right)^{(3)} = \right.$
$\left. = \dfrac{3!}{s^4} = \dfrac{6}{s^4}\right)$

$\Downarrow$

$(s^2+s-1)Y(s) = \dfrac{6}{s^4} + s + 1 = \dfrac{s^5+s^4+6}{s^4} \Rightarrow$

$$\boxed{Y(s) = \frac{s^5+s^4+6}{(s^2+s-1)s^4}}$$

4 $y''+6y'+5y = 12e^t,\ y(0)=-1,\ y'(0)=7$

$\mathcal{L}\{y'\} = sY(s) - y(0) = sY(s)+1$

$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s)+s-7$

$\Downarrow$

$\mathcal{L}\{y''+6y'+5y\} = (s^2+6s+5)Y(s) + \underbrace{s-7+6}_{s-1} = \mathcal{L}\{12e^t\} = \dfrac{12}{s-1} \Rightarrow$

$\underbrace{(s^2+6s+5)}_{(s+1)(s+5)}Y(s) = \dfrac{12}{s-1} - (s-1) = \dfrac{12-(s-1)^2}{s-1} = \dfrac{-s^2+2s+11}{s-1} \Rightarrow$

$$Y(s) = \frac{-s^2+2s+11}{(s-1)(s+1)(s+5)}$$

Find the inverse Laplace transform of $Y(s)$ by means of the partial fraction decomposition

$$\frac{-s^2+2s+11}{(s-1)(s+1)(s+5)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+5}$$

$$-s^2+2s+11 = A(s+1)(s+5) + B(s-1)(s+5) + C(s-1)(s+1)$$

To determine A plug in $s=1$: $-1+2+11 = A\cdot 2\cdot 6 \Rightarrow 12 = 12A \Rightarrow A=1$
To determine B plug in $s=-1$: $-1-2+11 = B\cdot(-2)\cdot 4 \Rightarrow -8=8B \Rightarrow B=-1$
To determine C plug in $s=-5$: $-25-10+11 = C(-6)(-4) \Rightarrow -24=24C \Rightarrow C=-1$

$\Downarrow$

$$Y(s) = \frac{-s^2+2s+11}{(s-1)(s+1)(s+5)} = \frac{1}{s-1} - \frac{1}{s+1} - \frac{1}{s+5} \Rightarrow$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\} = \boxed{e^{t}-e^{-t}-e^{-5t}}$$

5. a) $f(t) = t\cdot u_1(t) + (1-t)u_2(t) = (t-1)u_1(t) + u_1(t) - (t-2)u_2(t) + u_2(t) \Rightarrow$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{(t-1)u_1(t)\} - \mathcal{L}\{(t-2)u_2(t)\} + \mathcal{L}\{u_1(t)\} - \mathcal{L}\{u_2(t)\}$$

Recall that $\mathcal{L}\{t\} = \frac{1}{s^2} \Rightarrow$ by the propertion of translation in t

$$\mathcal{L}\{(t-1)u_1(t)\} = \frac{e^{-s}}{s^2}$$
$$\mathcal{L}\{(t-2)u_2(t)\} = \frac{e^{-2s}}{s^2}$$

Besides $\mathcal{L}\{u_1(t)\} = \frac{e^{-s}}{s}$, $\mathcal{L}\{u_2(t)\} = \frac{e^{-2s}}{s} \Rightarrow$

$$\mathcal{L}\{f(t)\} = \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2} + \frac{e^{-s}}{s} - \frac{e^{-2s}}{s} = \frac{s+1}{s^2}e^{-s} - \frac{s+1}{s^2}\cdot e^{-2s} =$$
$$= \boxed{\frac{s+1}{s^2}\left(e^{-s}-e^{-2s}\right)}$$

5 b) $\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t}$ $\Rightarrow$ by the property of translation in t

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s+2}\right\} = u_2(t)\, e^{-2(t-2)}$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-4s}}{s+2}\right\} = u_4(t)\, e^{-2(t-4)}$$

$\Downarrow$

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}-3e^{-4s}}{s+2}\right\} = \boxed{u_2(t)\, e^{-2(t-2)} - 3u_4(t)\, e^{-2(t-4)}}$$