Homework #8, MATH308, Solutions

Problem 1

$$AB = \begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 2-3 & 4-2 \\ -3+12 & -6+8 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 9 & 2 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} 2-6 & -1+8 \\ 6-6 & -3+8 \end{pmatrix} = \begin{pmatrix} -4 & 7 \\ 0 & 5 \end{pmatrix}$$

$$AB - BA = \begin{pmatrix} -1 & 2 \\ 9 & 2 \end{pmatrix} - \begin{pmatrix} -4 & 7 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 3 & -5 \\ 9 & -3 \end{pmatrix}$$

Remark This exercize demonstrates that in general $AB \neq BA$ i.e. the matrix multiplication is not a commutative operation.
The matrix $AB-BA$ is called the commutator of the matrices A and B and often is denoted by $[A,B]$.

Problem 2 (a)

Set $x_1 = u$, $x_2 = u'$ $\Rightarrow x_1' = x_2$ and $x_2' = u''$

$u'' + 3u' + 4u = \cos t \iff u'' = -3u' - 4u + \cos t \Rightarrow$

$$\begin{cases} x_1' = x_2 \\ x_2' = -4x_1 - 3x_2 + \cos t \end{cases}$$

(b) Set $x_1 = y$, $x_2 = y'$, $x_3 = y''$ $\Rightarrow x_1' = x_2$, $x_2' = x_3$ and $x_3' = y^{(3)}$

$y^{(3)} - 2y' + y = 0 \iff y^{(3)} = 2y' - y \Rightarrow x_3' = -x_1 + 2x_2$

So our system is

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = -x_1 + 2x_2 \end{cases}$$

Proplem 3

(a) $X' = \begin{pmatrix} 2 & 0 & -3 \\ 0 & 1 & 4 \\ 1 & 0 & 1 \end{pmatrix} X$ or, equivalently

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 2 & 0 & -3 \\ 0 & 1 & 4 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

(b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \sin t & e^t \\ \cos t & -e^t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \cos t \\ 0 \end{pmatrix}$

Problem 4 To answer this type of questions it is enough to check whether the Wronskian of the given set of solutions vanishes at some point t_0

(a) take $t_0 = 0$

$W(x^1, x^2)(0) = \begin{vmatrix} 1 & -2 \\ -2 & 4 \end{vmatrix} = 0 \Longrightarrow$

It is not a fundamental set of solutions.

(b) Take $t_0 = 0$

$$W(x^1, x^2, x^3) = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = 1+1=2 \neq 0 \Rightarrow$$

It is a fundamental set of solution.

Remark You may calculate the Wronskian for all t and check whether it vanishes or not (although it is usually longer calculations). The point is that if the Wronskian of solutions of a linear homogeneous differential equation vanishes at one point then it vanishes at any point and this fact is used in the solution above.

Problem 5 (a) The matrix of the system is

$$A = \begin{pmatrix} 1 & 3 \\ 12 & 1 \end{pmatrix}$$

i) Find the eigenvalues of A:

Characteristic polynomial is

$\det(A-\lambda I) = \lambda^2 - \operatorname{tr} A\,\lambda + \det A =$

$= \lambda^2 - (1+1)\lambda + (1-36) = \lambda^2 - 2\lambda - 35 = 0$

$D = 4 + 4\cdot 35 = 144 = 12^2$

$\lambda_1 = \frac{2+12}{2} = 7$

$\lambda_2 = \frac{2-12}{2} = -5$ $\Rightarrow$ 2 distinct real eigenvalues 7 and -5

ii) Find an eigenvector corresponding to $\lambda = 7$

$$(A-7I)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1-7 & 3 \\ 12 & 1-7 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -6 & 3 \\ 12 & -6 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow$$

$-6v_1 + 3v_2 = 0 \Rightarrow v_2 = 2v_1$

$12v_1 - 6v_2 = 0 \Rightarrow$ the second equation is obtained from the first one by multiplication by -2

Set $v_1 = 1 \Rightarrow v_2 = 2 \Rightarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector $\Rightarrow$

$e^{7t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is a solution of our sy

iii) Find an eigenvector corresponding to $\lambda = -5$

$$(A-(-5)I)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = (A+5I)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1+5 & 3 \\ 12 & 1+5 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} =$$

$$= \begin{pmatrix} 6 & 3 \\ 12 & 6 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{l} 6v_1 + 3v_2 = 0 \\ 12v_1 + 6v_2 = 0 \end{array}$$

$\Rightarrow$ the first equation, multiplied by 2

$6v_1 + 3v_2 = 0 \Rightarrow v_2 = -2v_1$.

Set $v_1 = 1 \Rightarrow v_2 = -2 \Rightarrow \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue -5. $\Rightarrow$

$e^{-5t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is a solution of our system

iv) From ii) and iii) $e^{7t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $e^{-5t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ constitute a fundamental set of solutions and the general solution is

$$\boxed{x(t) = c_1 e^{7t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-5t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}}$$

(B) Determine the constants C_1 and C_2 corresponding to the given initial conditions

$$x(0) = C_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Leftrightarrow$$

$$\begin{matrix} C_1 + C_2 = 1 \\ 2C_1 - 2C_2 = 1 \end{matrix} \xRightarrow{2\times Eq1 - Eq2} 4C_2 = 1 \Rightarrow C_2 = \frac{1}{4} \Rightarrow$$

$$C_1 = 1 - C_2 = 1 - \frac{1}{4} = \frac{3}{4}$$

$\Downarrow$

$$\boxed{x(t) = \frac{3}{4}e^{7t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{1}{4}e^{-5t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}}$$

(c) A solution

$$x(t) = c_1 e^{7t}\begin{pmatrix}1\\2\end{pmatrix} + c_2 e^{-5t}\begin{pmatrix}1\\-2\end{pmatrix}$$

approach 0 as $t \to +\infty \iff c_1 = 0$, i.e

$$x(t) = c_2 e^{-5t}\begin{pmatrix}1\\-2\end{pmatrix} \iff$$

$$x(0) = c_2\begin{pmatrix}1\\-2\end{pmatrix} \Rightarrow \boxed{\begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix} = \text{const}\begin{pmatrix}1\\-2\end{pmatrix}}$$

$\boxed{\text{i.e } \alpha_2 = -2\alpha_1}$

(d) A solution

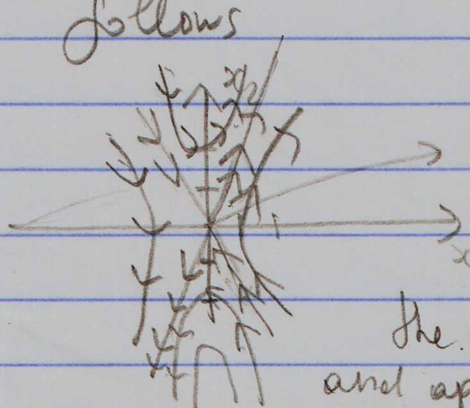
$$x(t) = c_1 e^{7t}\begin{pmatrix}1\\2\end{pmatrix} + c_2 e^{-5t}\begin{pmatrix}1\\-2\end{pmatrix}$$

approach 0 as $t \to -\infty \iff c_2 = 0$, i.e.

$$x(t) = c_1 e^{7t}\begin{pmatrix}1\\2\end{pmatrix} \iff x(0) = c_1\begin{pmatrix}1\\2\end{pmatrix} \Rightarrow$$

$$\boxed{\begin{pmatrix}\beta_1\\\beta_2\end{pmatrix} = \text{const}\begin{pmatrix}1\\2\end{pmatrix}}, \text{ i.e. } \boxed{\beta_2 = 2\beta_1}$$

<u>Remark</u> The phase portrait here looks as follows



0 is a saddle point

A solution approach 0 as $t \to \infty$ if and only if it starts on the eigenline of the negative eigenvalue -5 and approach 0 as $t \to -\infty$ if and only if it starts on the eigenline of the positive eigenvalue 7

Problem 6

a) The matrix of the system is

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 0 & 3 \\ 2 & 3 & 0 \end{pmatrix}$$

i) Find the eigenvalues of A

Characteristic equation is

$\det(A - \lambda I) = 0$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & 2 \\ 2 & -\lambda & 3 \\ 2 & 3 & -\lambda \end{vmatrix} =$$

$$= (1-\lambda)(\lambda^2-9) - 2(-2\lambda-6) + 2(6+2\lambda) =$$

$$= -(\lambda-1)(\lambda-3)(\lambda+3) + 4 \cdot 2(\lambda+3) = -((\lambda-1)(\lambda-3)-8)(\lambda+3) =$$

$$= -(\underbrace{\lambda^2-4\lambda-5}_{(\lambda+1)(\lambda-5)})(\lambda+3) = -(\lambda+1)(\lambda-5)(\lambda+3) = 0 \Rightarrow$$

We have 3 distinct eigenvalue

$\lambda = -3, -1, 5$

ii) Find an eigenvector corresponding to $\lambda = -3$

$$\Big(\underbrace{A-(-3)I}_{A+3I}\Big)\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 3 & 3 \\ 2 & 3 & 3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Use the Gauss elimination method:

$$\left(\begin{array}{ccc|c} 4 & 2 & 2 & 0 \\ 2 & 3 & 3 & 0 \\ 2 & 3 & 3 & 0 \end{array}\right) \xrightarrow[R_3 \leftrightarrow R_3 - R_2]{R_2 \leftrightarrow 2R_2 - R_1} \left(\begin{array}{ccc|c} 4 & 2 & 2 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[R_2 \to \frac{R_2}{4}]{R_1 \to \frac{R_1}{2}} \left(\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow$ $2v_1 + v_2 + v_3 = 0 \quad 2v_1 = -v_2 - v_3 = v_3 - v_3 = 0 \Rightarrow v_1 = 0$

$v_2 + v_3 = 0 \Rightarrow v_2 = -v_3$

Set $v_3 = 1 \Rightarrow v_2 = -1 \Rightarrow \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ is an eigenvector,

corresponding to the eigenvalue $\lambda = -3 \Rightarrow$

$e^{-3t}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ is a solution of our equation

(iii) Find an eigenvector corresponding to the eigenvalue $\lambda = -1$

$$\Big(\underbrace{A-(-1)I}_{A+I}\Big)\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Use the Gauss elimination method:

$$\left(\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 2 & 1 & 3 & 0 \\ 2 & 3 & 1 & 0 \end{array}\right) \xrightarrow[R_3 \leftrightarrow R_3 - R_1]{R_2 \leftrightarrow R_2 - R_1} \left(\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right) \xrightarrow[R_1 \to R_1/2]{R_3 \to R_3 + R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow$$

$$V_1 + V_2 + V_3 = 0$$
$$-V_2 + V_3 = 0$$

Set $v_3 = 1 \Rightarrow V_2 = 1 \Rightarrow V_1 = -V_2 - V_3 = -2 \Rightarrow$

$\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue

$\lambda = -1 \Rightarrow e^{-t}\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ is a solution of our system

(iv) Find an eigenvector corresponding to the eigenvalue

$\lambda = 5$

$$(A - 5I)\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1-5 & 2 & 2 \\ 2 & -5 & 3 \\ 2 & 3 & -5 \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} -4 & 2 & 2 \\ 2 & -5 & 3 \\ 2 & 3 & -5 \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = 0$$

Use the Gauss elimination method:

$$\left(\begin{array}{ccc|c} -4 & 2 & 2 & 0 \\ 2 & -5 & 3 & 0 \\ 2 & 3 & -5 & 0 \end{array}\right) \xrightarrow[R_3 \leftrightarrow 2R_3 + R_1]{R_2 \leftrightarrow 2R_2 + R_1} \left(\begin{array}{ccc|c} -4 & 2 & 2 & 0 \\ 0 & -8 & 8 & 0 \\ 0 & 8 & -8 & 0 \end{array}\right) \xrightarrow[R_2 \to \frac{R_2}{8},\ R_1 \to \frac{R_1}{2}]{R_3 \leftrightarrow R_3 + R_2}$$

$$\to \left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} -2V_1 + V_2 + V_3 = 0 \\ -V_2 + V_3 = 0 \end{array}$$

Set $v_3 = 1 \Rightarrow v_2 = 1 \Rightarrow -2V_1 = -2 \Rightarrow V_1 = 1 \Rightarrow$

$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector, corresponding to the eigenvalue

$\lambda = 5 \Rightarrow e^{5t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is a solution of our system

v) Combining the results of item ii) – iv) we get that the general solution is

$$\boxed{x(t) = C_1 e^{-3t}\begin{pmatrix}0\\-1\\1\end{pmatrix} + C_2 e^{-t}\begin{pmatrix}-2\\1\\1\end{pmatrix} + C_3 e^{5t}\begin{pmatrix}1\\1\\1\end{pmatrix}}$$

b) Determine constants $C_1, C_2,$ and C_3 for which $x(0) = \begin{pmatrix}3\\-2\\2\end{pmatrix}$:

$$x(0) = C_1\begin{pmatrix}0\\-1\\1\end{pmatrix} + C_2\begin{pmatrix}-2\\1\\1\end{pmatrix} + C_3\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}3\\-2\\2\end{pmatrix} \Leftrightarrow$$

$$\begin{pmatrix}0 & -2 & 1\\-1 & 1 & 1\\1 & 1 & 1\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = \begin{pmatrix}3\\-2\\2\end{pmatrix}$$

Use the Gauss eliminination method to find C_1, C_2, C_3.

$$\left(\begin{array}{ccc|c}0 & -2 & 1 & 3\\-1 & 1 & 1 & -2\\1 & 1 & 1 & 2\end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c}-1 & 1 & 1 & -2\\0 & -2 & 1 & 3\\1 & 1 & 1 & 2\end{array}\right) \xrightarrow{R_3 \to R_3 + R_1} \left(\begin{array}{ccc|c}-1 & 1 & 1 & -2\\0 & -2 & 1 & 3\\0 & 2 & 2 & 0\end{array}\right) \xrightarrow{R_3 \to R_3 + R_2}$$

$$\left(\begin{array}{ccc|c}-1 & 1 & 1 & -2\\0 & -2 & 1 & 3\\0 & 0 & 3 & 3\end{array}\right) \Rightarrow \begin{aligned} -C_1 + C_2 + C_3 &= -2 \Rightarrow -C_1 + (-1) + 1 = -2 \Rightarrow C_1 = 2\\ -2C_2 + C_3 &= 3 \Rightarrow -2C_2 + 1 = 3 \Rightarrow \boxed{C_2 = -1}\\ 3C_3 &= 3 \Rightarrow \boxed{C_3 = 1}\end{aligned}$$

$$\Rightarrow \boxed{x(t) = 2e^{-3t}\begin{pmatrix}0\\-1\\1\end{pmatrix} - e^{-t}\begin{pmatrix}-2\\1\\1\end{pmatrix} + e^{5t}\begin{pmatrix}1\\1\\1\end{pmatrix}}$$