1. Determine if the vector field $\mathbf{F}(x, y) = \langle 12xy + 28x^3y^7 + 2013, 49x^4y^6 + 6x^2 - 24y^2\rangle$ is conservative or not. If it is conservative, find its potential.

$$P = 12xy + 28x^3y^7 + 2013 \Rightarrow \frac{\partial P}{\partial y} = 12x + 196x^3y^6$$

$$Q = 49x^4y^6 + 6x^2 - 24y \Rightarrow \frac{\partial Q}{\partial x} = 196x^3y^6 + 12x$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Rightarrow \text{The field is conservative}$$

To find a potential $f(x, y)$: $\nabla f = \vec{F}(x, y) \Rightarrow$

$$f_x = 12xy + 28x^3y^7 + 2013 \Rightarrow f(x,y) = \int (12xy + 28x^3y^7 + 2013)\,dx$$

$$\Downarrow$$

$$f_y = 49x^4y^6 + 6x^2 - 24y \qquad f(x,y) = 6x^2y + 7x^4y^7 + 2013x + C(y)$$

$$\Downarrow$$

$$49x^4y^6 + 6x^2 - 24y = \frac{\partial}{\partial y}\left[6x^2y + 7x^4y^7 + 2013x + C(y)\right]$$

$$49x^4y^6 + 6x^2 - 24y = 6x^2 + 49x^4y^6 + C'(y)$$

$$C'(y) = -24y \Rightarrow C(y) = -12y^2 + C$$

$$f(x,y) = 6x^2y + 7x^4y^7 + 2013x - 12y^2 + C$$

2. Evaluate $\int_C \nabla f \cdot d\mathbf{r}$ for $f(x, y, z) = xyz^2$ and C is given by $\mathbf{r}(t) = \langle t, t^2 + 1, t + 2 \rangle$, $1 \leq t \leq 2$.

$\vec{r}(1) = \langle 1, 2, 3 \rangle$, $\vec{r}(2) = \langle 2, 5, 4 \rangle$

By Fundamental Theorem for Line Integral:

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(2)) - f(\vec{r}(1)) = f(2, 5, 4) - f(1, 2, 3)$$

$$= 2 \cdot 5 \cdot 4^2 - 1 \cdot 2 \cdot 3^2 = 160 - 18 = \boxed{142}$$

3. Find the potential function for the vector field

$$\mathbf{F}(x,y,z) = \left\langle 2x\ln(y^2z), \frac{2x^2}{y} - 27y^2z^4, \frac{x^2}{z} - 36y^3z^3 \right\rangle.$$

Potential $f(x,y,z)$: $\nabla f = \vec{F} \Rightarrow$

$f_x = 2x\ln(y^2z) \Rightarrow f = \int 2x\ln(y^2z)\,dx$

$f = x^2\ln(y^2z) + C(y,z)$

$\Downarrow$

$f_y = \frac{2x^2}{y} - 27y^2z^4$

$f_y = \frac{2x^2yz}{y^2z} + C_y(y,z)$

$f_y = \frac{2x^2}{y} + C_y(y,z)$

$\Downarrow$

$\frac{2x^2}{y} - 27y^2z^4 = \frac{2x^2}{y} + C_y(y,z)$

$C_y(y,z) = -27y^2z^4 \Rightarrow C(y,z) = -9y^3z^4 + K(z)$

$f_z = \frac{x^2}{z} - 36y^3z^3$

$f = x^2\ln(y^2z) - 9y^3z^4 + K(z)$

$f_z = \frac{x^2y^2}{y^2z} - 36y^3z^3 + K'(z)$

$f_z = \frac{x^2}{z} - 36y^3z^3 + K'(z)$

$\Downarrow$

$\frac{x^2}{z} - 36y^3z^3 = \frac{x^2}{z} - 36y^3z^3 + K'(z) \Rightarrow K'(z) = 0$

$K(z) = C$

Finally,

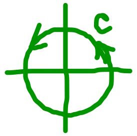
$f(x,y,z) = x^2\ln(y^2z) - 9y^3z^4 + C$

4. Consider the line integral $\int_C \langle 3-8y, 4x+y\rangle \cdot \mathbf{dr}$, where C is the positively oriented circle of radius 1 centered at the origin. Evaluate the integral

(a) directly;

Parameterize C: $x^2+y^2=1$

$x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$



$dx = -\sin t\, dt$. $dy = \cos t\, dt$

$$\int_C \langle 3-8y, 4x+y\rangle \cdot dr = \int_C (3-8y)\,dx + (4x+y)\,dy$$

$$= \int_0^{2\pi} \left[(3-8\sin t)(-\sin t) + (4\cos t + \sin t)\cos t\right] dt$$

$$= \int_0^{2\pi} \left[-3\sin t + 8\sin^2 t + 4\cos^2 t + \sin t\cos t\right] dt$$

$$= \int_0^{2\pi} -3\sin t + 4\sin^2 t + \underbrace{4\sin^2 t + 4\cos^2 t}_{4} + \sin t\cos t\,] dt$$

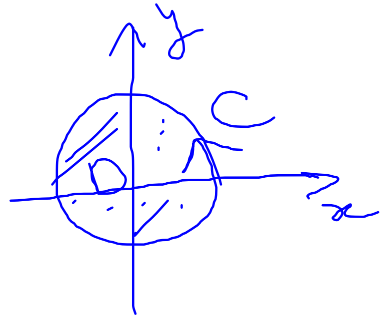
$$= -3\int_0^{2\pi} \sin t\,dt + 4\int_0^{2\pi} \sin^2 t\,dt + 4\int_0^{2\pi} dt + \int_0^{2\pi} \sin t\cos t\,dt$$

$$= -3\cdot 0 + 2\int_0^{2\pi}(1-\cos 2t)\,dt + 4\cdot 2\pi + \frac{\sin^2 t}{2}\Big|_0^{2\pi}$$

$$= 2\left[2\pi - \frac{\sin 2t}{2}\Big|_0^{2\pi}\right] + 8\pi + \frac{1}{2}(0-0)$$

$$= 4\pi - 0 + 8\pi = 12\pi$$

(b) using Green's theorem.



$P = 3 - 8y$, $Q = 4x + y$

$\frac{\partial P}{\partial y} = -8$ $\frac{\partial Q}{\partial x} = 4$

$$\int_C \langle 3-8y,\ 4x+y \rangle \cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$= \iint_D (4-(-8))\, dA = 12 \iint_D dA = 12 \cdot (\text{Area } D)$$

$$= 12 \cdot \pi \cdot 1^2 = 12\pi$$