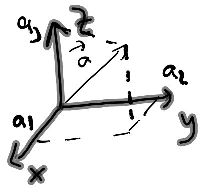
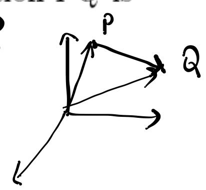
11.2: Vectors and the Dot Product in Three Dimensions

DEFINITION 1. *A* **3-dimensional vector** *is an ordered triple* $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$

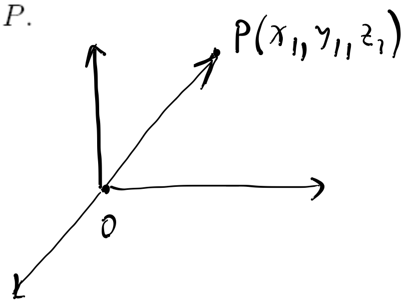


Given the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, the vector $\mathbf{a}$ with representation $\overrightarrow{PQ}$ is

$$\overrightarrow{PQ} = \mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle = \overrightarrow{OQ} - \overrightarrow{OP}$$



The representation of the vector that starts at the point $O(0, 0, 0)$ and ends at the point $P(x_1, y_1, z_1)$ is called the **position** vector of the point P.



$$\overrightarrow{OP} = \langle x_1, y_1, z_1 \rangle$$
$$\parallel$$
$$\langle x_1 - 0, y_1 - 0, z_1 - 0 \rangle$$

EXAMPLE 2. *Find the vector represented by the directed line segment with the initial point $A(1,2,3)$ and terminal point $B(3,2,-1)$. What is the position vector of the point A?* $= \vec{OA} = \langle 1,2,3 \rangle$

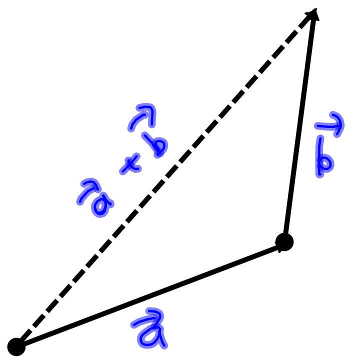
$$\vec{AB} = \vec{OB} - \vec{OA} = \langle 3,2,-1 \rangle - \langle 1,2,3 \rangle$$

$$= \langle 3-1, 2-2, -1-3 \rangle = \langle 2,0,-4 \rangle$$

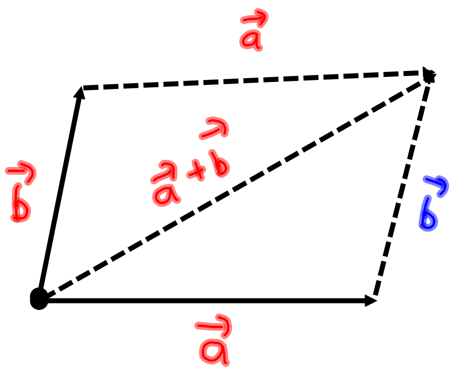
Vector Arithmetic: Let $a = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$.

- Scalar Multiplication: $\alpha\mathbf{a} = \langle \alpha a_1, \alpha a_2, \alpha a_3 \rangle$, $\alpha \in \mathbb{R}$.
- Addition: $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$

TRIANGLE LAW



PARALLELOGRAM LAW



Two vectors **a** and **b** are parallel if one is a scalar multiple of the other, i.e. there exists $\alpha \in \mathbb{R}$ s.t. $\mathbf{b} = \alpha\mathbf{a}$. Equivalently:

$$\mathbf{a} \| \mathbf{b} \quad \Leftrightarrow \quad \vec{b} = \alpha \vec{a}$$

$\vec{a} = \langle a_1, a_2, a_3 \rangle$ $\quad\quad$ $\vec{b} = \langle b_1, b_2, b_3 \rangle$

$\langle b_1, b_2, b_3 \rangle = \alpha \langle a_1, a_2, a_3 \rangle = \langle \alpha a_1, \alpha a_2, \alpha a_3 \rangle$

$\vec{a} \| \vec{b}$

$b_1 = \alpha a_1$

$_2 = \alpha a_2$

$b_3 = \alpha a_3$

$\Rightarrow$ $\boxed{\text{if } a_1 a_2 a_3 \neq 0 \quad \frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3}}$

For example, $\vec{a} = \langle 1, 2, 3 \rangle$, $\vec{b} = \langle -\frac{1}{2}, -1, -\frac{3}{2} \rangle$

Then $\frac{1}{-1/2} = \frac{2}{-1} = \frac{3}{-3/2} = -2$ $\quad \Rightarrow$

$\Rightarrow \vec{a} \| \vec{b}$

$\vec{c} = \langle 1, 0, 3 \rangle \quad \Rightarrow \quad \vec{a} \nparallel \vec{c}$

The magnitude or length of $a = \langle a_1, a_2, a_3 \rangle$:

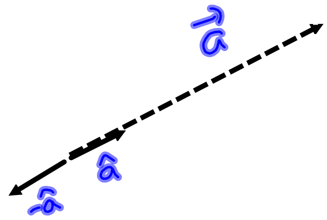
$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

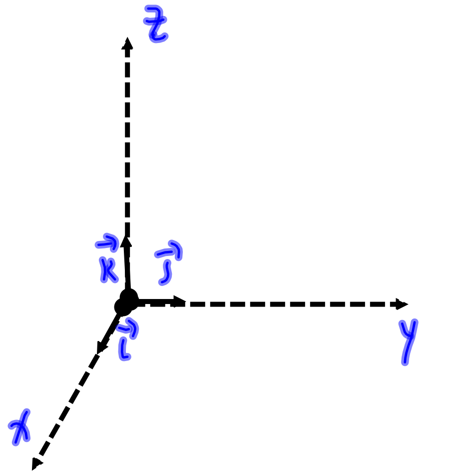
`Zero vector:` $\mathbf{0} = \langle 0, 0, 0 \rangle$, $|\vec{0}| = 0$.

Note that $|\mathbf{a}| = 0 \Leftrightarrow \mathbf{a} = \mathbf{0}$.

`Unit vector:` $\hat{\mathbf{a}} = \dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{\langle a_1, a_2, a_3 \rangle}{\sqrt{a_1^2+a_2^2+a_3^2}} = \left\langle \dfrac{a_1}{\sqrt{}}, \dfrac{a_2}{\sqrt{}}, \dfrac{a_3}{\sqrt{}} \right\rangle$

$|\hat{a}| = 1$





Standard Basis Vectors:

$\mathbf{i} = \langle 1, 0, 0 \rangle$

$\mathbf{j} = \langle 0, 1, 0 \rangle$

$\mathbf{k} = \langle 0, 0, 1 \rangle$

Note that $|\mathbf{i}| = |\mathbf{j}| = |\mathbf{k}| = 1$.

We have:

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle = \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle$$

$$= a_1 \langle 1, 0, 0 \rangle + a_2 \langle 0, 1, 0 \rangle + a_3 \langle 0, 0, 1 \rangle$$

$$= a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$$

$$\langle 2, 3, -7 \rangle = 2\vec{i} + 3\vec{j} - 7\vec{k}$$

EXAMPLE 3. *Given* $\mathbf{a} = \langle 1, 0, -3\rangle$ *and* $\mathbf{b} = \langle 3, 1, 2\rangle$. *Find*

(a) $|\mathbf{b} - \mathbf{a}|$.

$$\vec{b} - \vec{a} = \langle 3,1,2\rangle - \langle 1,0,-3\rangle = \langle 3-1, 1-0, 2-(-3)\rangle$$

$$|\vec{b} - \vec{a}| = |\langle 2, 1, 5\rangle| = \sqrt{2^2 + 1^2 + 5^2} = \sqrt{4 + 1 + 25} = \sqrt{30}$$

(b) *a unit vector that has the same direction as* $\mathbf{b}$.

$$\hat{b} = \frac{\vec{b}}{|\vec{b}|}$$

$$|\vec{b}| = \sqrt{3^2 + 1^2 + 2^2} = \sqrt{9 + 1 + 4} = \sqrt{14}$$

$$\hat{b} = \frac{\vec{b}}{|\vec{b}|} = \frac{\langle 3,1,2\rangle}{\sqrt{14}} = \left\langle \frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}} \right\rangle$$

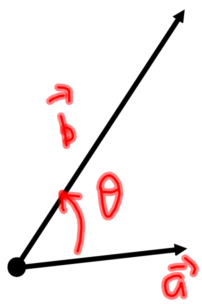
OR

$$\hat{b} = \frac{3}{\sqrt{14}}\hat{i} + \frac{1}{\sqrt{14}}\vec{j} + \frac{2}{\sqrt{14}}\vec{k}$$

Dot Product of two nonzero vectors **a** and **b** is the $\boxed{\text{NUMBER:}}$

The same

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\cdot|\mathbf{b}|\cos\theta,$$



where θ is the angle between **a** and **b**, $0 \le \theta \le \pi$.

If $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$ then $\mathbf{a}\cdot\mathbf{b} = 0$.

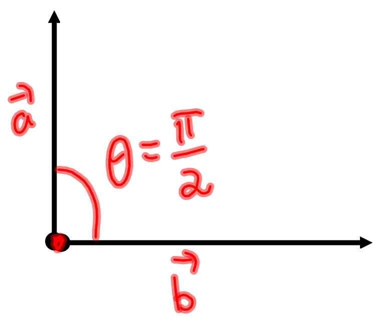
Component Formula for dot product of $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$:

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

If θ is the *angle* between two nonzero vectors **a** and **b**, then

$$\boxed{\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\cdot|\mathbf{b}|}} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\,\sqrt{b_1^2 + b_2^2 + b_3^2}}$$

DEFINITION 4. *Two nonzero vectors* **a** *and* **b** *are called* **perpendicular** *or orthogonal if the angle between them is* $\theta = \pi/2$.



$$\vec{a} \perp \vec{b} \iff \theta = \frac{\pi}{2}$$

$$\vec{a} \perp \vec{b} \Rightarrow \theta = \frac{\pi}{2}$$

$$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\theta = |\vec{a}| \cdot |\vec{b}| \underbrace{\cos\frac{\pi}{2}}_{=0} = 0$$

If $\vec{a} \cdot \vec{b} = 0 \Rightarrow \underbrace{|\vec{a}|}_{\neq 0} \cdot \underbrace{|\vec{b}|}_{\neq 0} \cos\theta = 0 \Rightarrow \cos\theta = 0$

$\theta = \frac{\pi}{2}$

Conclusion $\boxed{\vec{a} \perp \vec{b} \iff \vec{a} \cdot \vec{b} = 0}$

EXAMPLE 6. *For what value(s) of c are the vectors* $\underbrace{c\mathbf{i} + 2\mathbf{j} + \mathbf{k}}_{\vec{a}}$ *and* $\underbrace{4\mathbf{i} + 3\mathbf{j} + c\mathbf{k}}_{\vec{b}}$ *orthogonal?*

$$\vec{a} \perp \vec{b} \iff \vec{a} \cdot \vec{b} = 0$$

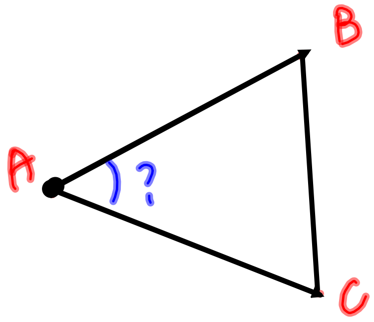
$$\langle c, 2, 1 \rangle \cdot \langle 4, 3, c \rangle = 0$$

$$4c + 2 \cdot 3 + 1 \cdot c = 0$$

$$5c + 6 = 0$$

$$c = \boxed{-\frac{6}{5}}$$

EXAMPLE 7. *The points $A(6,-1,0)$, $B(-3,1,2)$, $C(2,4,5)$ form a triangle. Find angle at A.*



$\measuredangle A = \measuredangle BAC = \measuredangle \vec{AB}, \vec{AC}$

$$\cos\measuredangle A = \frac{\vec{AB}\cdot\vec{AC}}{|\vec{AB}|\cdot|\vec{AC}|}$$

$\vec{AB} = \vec{OB} - \vec{OA} = \langle -3,1,2\rangle - \langle 6,-1,0\rangle = \langle -3-6, 1-(-1), 2-0\rangle$

$\vec{AB} = \langle -9, 2, 2\rangle$

$\vec{AC} = \langle 2,4,5\rangle - \langle 6,-1,0\rangle = \langle 2-6, 4-(-1), 5-0\rangle = \langle -4,5,5\rangle$

$\vec{AB}\cdot\vec{AC} = \langle -9,2,2\rangle\cdot\langle -4,5,5\rangle = -9\cdot(-4)+2\cdot 5+2\cdot 5 = 56$

$|\vec{AB}| = |\langle -9,2,2\rangle| = \sqrt{(-9)^2+2^2+2^2} = \sqrt{81+4+4} = \sqrt{89}$

$|\vec{AC}| = |\langle -4,5,5\rangle| = \sqrt{4^2+5^2+5^2} = \sqrt{66}$

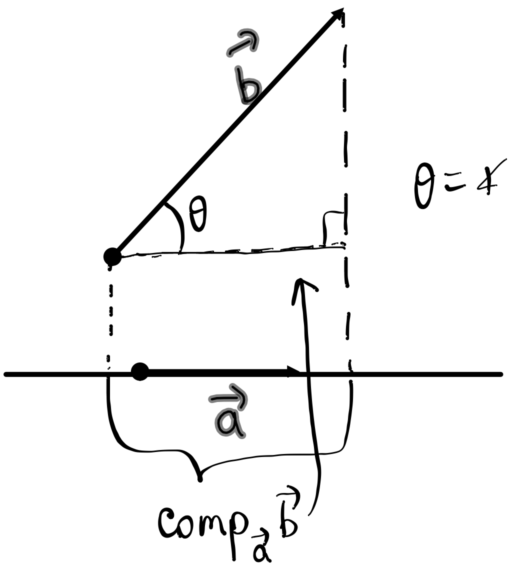
Plugin:

$$\cos\measuredangle A = \frac{56}{\sqrt{89}\cdot\sqrt{66}}$$

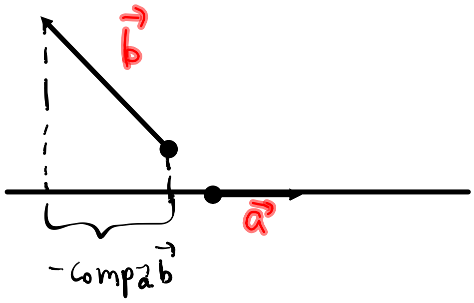
$$\measuredangle A = \arccos\frac{56}{\sqrt{89\cdot 66}} \approx 43^\circ$$

Projections:

- Scalar projection of vector b onto vector a: $\boxed{\mathtt{comp}_{\mathbf{a}}\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}}$



$\theta = \measuredangle \vec{a}, \vec{b}$

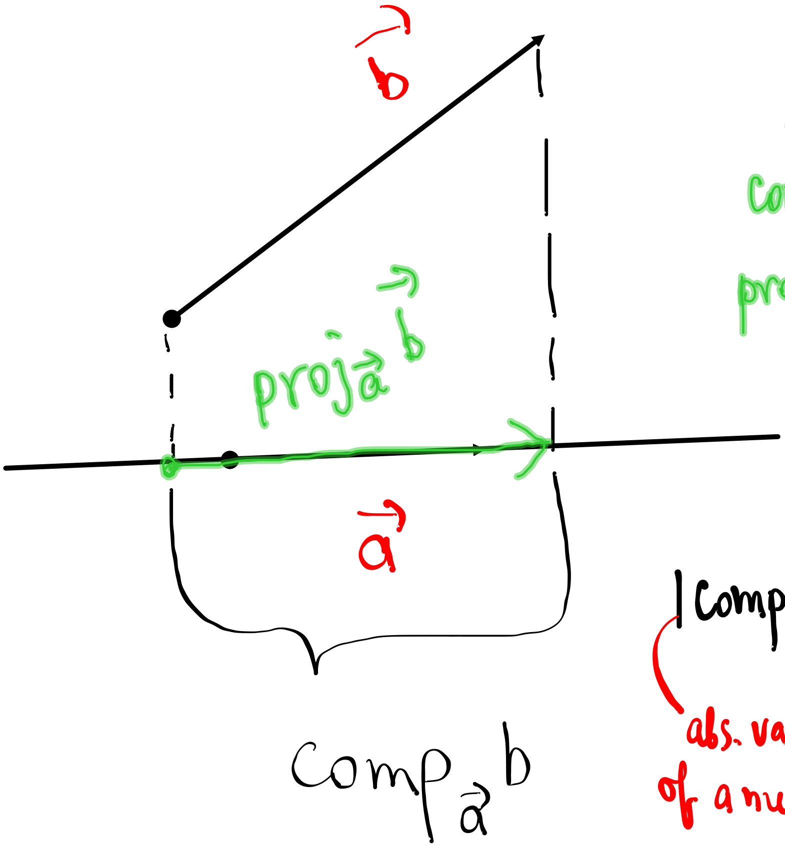


$$\mathrm{comp}_{\vec{a}}\vec{b} = |\vec{b}|\cos\theta = \cancel{|\vec{b}|}\cdot\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot\cancel{|\vec{b}|}} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$$

- <u>Vector projection</u> of vector b onto vector a:

$$\mathrm{proj}_{\mathbf a}\mathbf b = \underbrace{\left(\frac{\mathbf a\cdot\mathbf b}{|\mathbf a|}\right)}_{\parallel}\frac{\mathbf a}{|\mathbf a|} = \frac{(\vec a\cdot\vec b)\,\vec a}{|\vec a|^2}$$

$$(\mathrm{comp}_{\vec a}\vec b)\cdot\hat a$$



Note
$\mathrm{comp}_{\vec a}\vec b$ is a number
$\mathrm{proj}_{\vec a}\vec b$ is a vector

$$|\mathrm{comp}_{\vec a}\vec b| = |\mathrm{proj}_{\vec a}\vec b|$$

abs. value of a number

magnitude of a vector

EXAMPLE 8. *Find the scalar and vector projections of* $\underbrace{\langle 2, -2-1\rangle}_{\vec{u}}$ *onto* $\underbrace{\langle 3, 3, 4\rangle}_{\vec{v}}$.

$$\text{comp}_{\vec{v}}\,\vec{u} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|} = \frac{\langle 2,-2,-1\rangle\cdot\langle 3,3,4\rangle}{|\langle 3,3,4\rangle|}$$

$$= \frac{6-6-4}{\sqrt{9+9+16}} = -\frac{4}{\sqrt{34}}$$

$$\text{proj}_{\vec{v}}\,\vec{u} = \text{Comp}_{\vec{v}}\,\vec{u}\cdot\frac{\vec{v}}{|\vec{v}|} = -\frac{4}{\sqrt{34}}\cdot\frac{\langle 3,3,4\rangle}{\sqrt{34}}$$

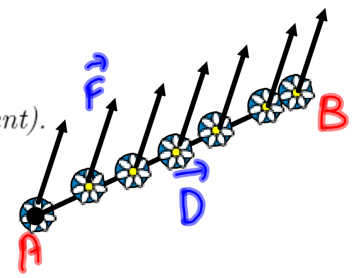
$$= -\frac{4}{34}\langle 3,3,4\rangle = -\frac{2}{17}\langle 3,3,4\rangle$$

$$\text{proj}_{\vec{v}}\,\vec{u} = \left\langle -\frac{6}{17}, -\frac{6}{17}, -\frac{8}{17}\right\rangle$$

DEFINITION 8. *The* **work** *done by a force* $\mathbf{F}$ *in moving and object from point* A *to point* B *is given by*

$$W = \mathbf{F} \cdot \mathbf{D}$$

where $\mathbf{D} = \overrightarrow{AB}$ *is the distance the object has moved (or displacement).*



EXAMPLE 9. *A force is given by a vector* $\mathbf{F} = \mathbf{i} - \mathbf{j} + 5\mathbf{k}$ *and moves a particle from the point* $P(1, 2, 0)$ *to the point* $Q(2, 3, 5)$*. Find the work done.*

$$\vec{D} = \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \langle 2,3,5 \rangle - \langle 1,2,0 \rangle = \langle 1,1,5 \rangle$$

$$W = \vec{F} \cdot \vec{D} = \langle 1,-1,5 \rangle \cdot \langle 1,1,5 \rangle = 1 - 1 + 25 = \boxed{25}$$