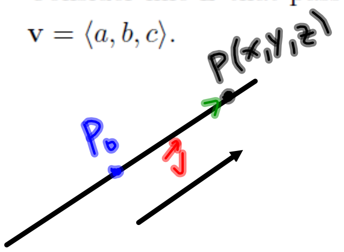
11.4: Equations of lines and planes

Lines

Lines determined by a point and a vector

Consider line L that passes through the point $P_0(x_0, y_0, z_0)$ and is parallel to the nonzero vector $\mathrm{v} = \langle a, b, c\rangle$.



$\overrightarrow{P_0P} \parallel \vec{v}$

There exists a real constant t such that

$\overrightarrow{P_0P} = t\vec{v}$

$\langle x-x_0, y-y_0, z-z_0\rangle = t\langle a, b, c\rangle$

$$\left.\begin{aligned} x - x_0 &= ta \\ y - y_0 &= tb \\ z - z_0 &= tc \end{aligned}\right\}$$

Parametric equations of the line:

point on line $\quad = \vec{v}$

$$\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned}$$

EXAMPLE 1. *Find parametric equations of the line*

(a) *passing through the point* $\overset{x_0, y_0, z_0}{(3, -4, 1)}$ *and parallel to* $\mathbf{v} = \overset{<a, b, c>}{\langle 7, 0, -1 \rangle}$

$x = 3 + 7t, \quad y = -4 + 0t, \quad z = 1 + (-1)\cdot t$

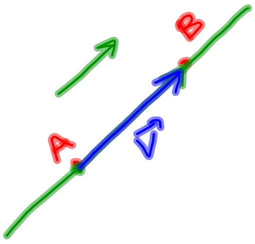
or $x = 3 + 7t, \quad y = -4, \quad z = 1 - t$

(b) *passing through the origin and parallel to* $\mathbf{v} = \langle 5, 5, 5 \rangle$

$x = 0 + 5t, \quad y = 0 + 5t, \quad z = 0 + 5t$

or $x = 5t, \quad y = 5t, \quad z = 5t$

EXAMPLE 2. *Consider the line L that passes through the points* $A(1, 1, 1)$ *and* $B(2, 3, -2)$. *Find points at that L intersects the yz-plane.*



Find param. equation of the line L.

$\vec{v} = \overrightarrow{BA} = \langle 2-1, 3-1, -2-1 \rangle = \langle 1, 2, -3 \rangle$

$x = 1 + 1t$
$y = 1 + 2t$
$z = 1 + (-3)t$

$\Rightarrow$ $\boxed{\begin{array}{l} x = 1 + t \\ y = 1 + 2t \\ z = 1 - 3t \end{array}}$

$L \cap$ yz-plane $\Rightarrow x = 0 \Rightarrow 1 + t = 0 \Rightarrow \boxed{t = -1}$

yz-plane $\Updownarrow$ $x = 0$

$y = 1 + 2\cdot(-1) = -1$

$z = 1 - 3\cdot(-1) = 4$

$\boxed{(0, -1, 4)}$

Symmetric equations of the line: If $abc \neq 0$ then

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c} = t \Leftarrow \boxed{\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned}}$$

If, for example, $a = 0$ then the symmetric equations have the form:

$$x = x_0, \quad \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

EXAMPLE 3. *Find symmetic equations of lines from Example 1.*

(a) $x = 3 + 7t$, $y = -4$, $z = 1 - t$

$$t = \boxed{\frac{x-3}{7} = \frac{z-1}{-1}, \quad y = -4}$$

(b) $x = 5t$, $y = 5t$, $z = 5t$

$$\boxed{\frac{x-0}{5} = \frac{y-0}{5} = \frac{z-0}{5}}$$

Vector equation of the line:

$$\mathbf{r}(t) = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle$$

where $P_0(x_0, y_0, z_0)$ is a given point on the line and $\mathbf{v} = \langle a, b, c \rangle$ is some vector which is parallel to the line, t is a parameter, $-\infty < t < \infty$.

$\vec{r}(t)$ $\langle x_0, y_0, z_0 \rangle$ $\vec{v} = \langle a, b, c \rangle$

$$\begin{aligned} x(t) &= x_0 + at \\ y(t) &= y_0 + bt \\ z(t) &= z_0 + ct \end{aligned}$$

$\vec{r}(t) = \langle x_0, y_0, z_0 \rangle + t\vec{v}$

EXAMPLE 4. *Find vector equation of the line that passes through the points* $P(1, 1, -4)$ *and* $Q(0, 3, -4)$.

$\vec{v} = \overrightarrow{PQ} = \langle 0-1, 3-1, -4-(-4) \rangle = \langle -1, 2, 0 \rangle$

$\boxed{\vec{r}(t) = \langle 1, 1, -4 \rangle + t \langle -1, 2, 0 \rangle}$

EXAMPLE 5. *Determine whether the lines*

$\vec{v}_1 = \langle 1, 3, -1 \rangle$

$$L_1: \underbrace{x-1}_{1} = \frac{y+2}{3} = \frac{z-4}{-1} \qquad P_1(1,-2,4)$$

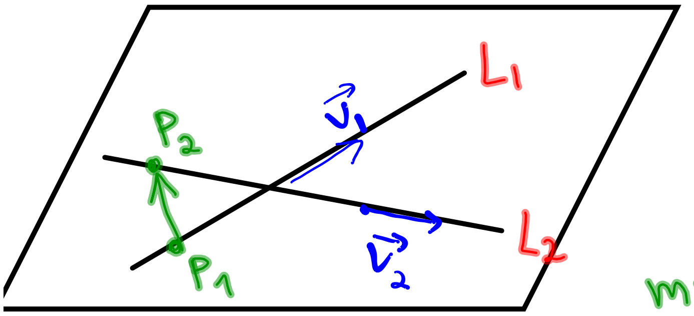
and

$\vec{v}_2 = \langle 2, 1, 4 \rangle$

$$L_2: \quad x = 2t, \quad y = 3+t, \quad z = -3+4t$$

are parallel, skew, or intersecting. $\quad P_2(0,3,-3)$

$$\vec{v}_1 \nparallel \vec{v}_2 \Rightarrow L_1 \nparallel L_2$$



If L_1 and L_2 are intersecting then $\vec{v}_1$, $\vec{v}_2$ & $\overrightarrow{P_1P_2} = \langle -1, 5, -7 \rangle$ must be coplanar, which means their scalar triple product must be zero:

$$\vec{v}_1 \cdot (\vec{v}_2 \times \overrightarrow{P_1P_2}) = \begin{vmatrix} 1 & 3 & -1 \\ 2 & 1 & 4 \\ -1 & 5 & -7 \end{vmatrix} = 1 \cdot (-7-20) - 3(-14+4) + (-1)(10+1)$$

$$= -27 + 30 - 11 \neq 0$$

$$\boxed{L_1 \,\&\, L_2 \text{ are skew}}$$

Line segments

How to find parametric equation of a line segment:

1. *Find parametric equation for the entire line;*
2. *restrict the parameter appropriately so that only the desired segment is generated.*

EXAMPLE 6. *Find parametric equations describing the line segment joining the points* $M(1, 2, 3)$ *and* $N(3, 2, 1)$.

$$\vec{V} = \overrightarrow{MN} = \langle 3-1, 2-2, 1-3 \rangle = \langle 2, 0, -2 \rangle$$

Line through M & N

$$x = 1 + 2t$$
$$y = 2 + 0t \Rightarrow$$
$$z = 3 + (-2)t$$

$$\boxed{\begin{array}{l} x = 1 + 2t \\ y = 2 \\ z = 3 - 2t \\ 0 \le t \le 1 \end{array}}$$

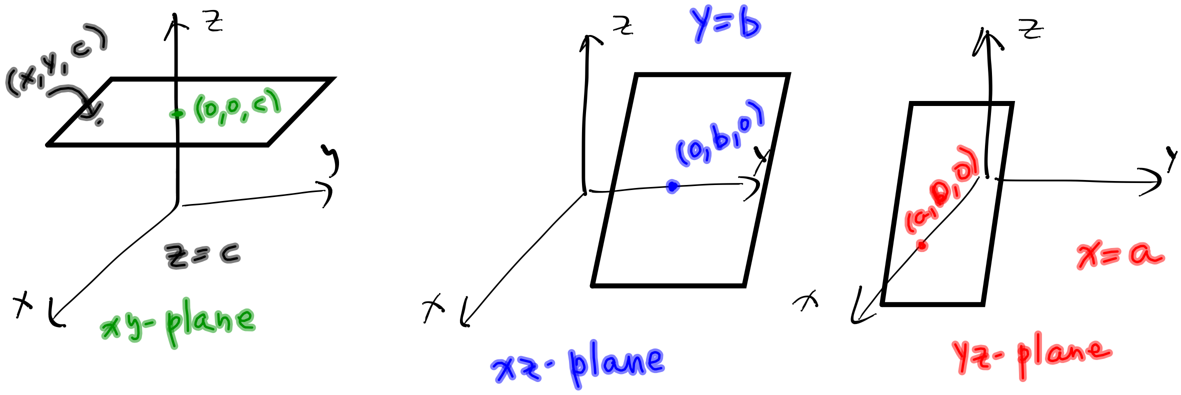
Line segment $\overline{MN}$

M $P(5, 2, -1)$

MP $\to$ $0 \le t \le 2$

Planes

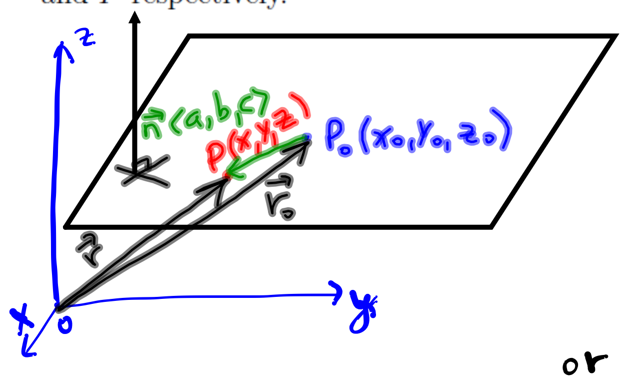
Planes parallel to the coordinate planes:



Planes determined by a point and a normal vector

A plane in $\mathbb{R}^3$ is uniquely determined by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector $\mathbf{n} = (a, b, c)$ that is orthogonal to the plane. This vector is called a **normal vector**.

Assume that $P(x, y, z)$ is any point in the plane. Let $\mathbf{r}_0$ and $\mathbf{r}$ be the position vectors for P_0 and P respectively.



$\vec{r} = \overrightarrow{OP} = \langle x, y, z\rangle$

$\vec{r}_0 = OP_0 = \langle x_0, y_0, z_0\rangle$

$\vec{n} \perp \overrightarrow{P_0P} = \vec{r} - \vec{r}_0$

$\Downarrow$

$\vec{n} \cdot \overrightarrow{P_0P} = 0$

or $\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$

Vector equation of the plane: $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0 \quad \Leftrightarrow \quad \mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0.$

$\langle a, b, c\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0$

$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

Scalar equation of plane:

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.}$$

$ax - ax_0 + by - by_0 + cz - cz_0 = 0$

$ax + by + cz = ax_0 + by_0 + cz_0 = d$

Often this will be written as a **linear equation** in x, y, z,

$$\boxed{ax + by + cz = d}$$

where $d = ax_0 + by_0 + cz_0$.

EXAMPLE 7. *Determine the equation of the plane through the point* $(1, 2, 1)$ *and orthogonal to vector* $\langle 2, 3, 4\rangle$. *Find the intercepts and sketch the plane.*

$\perp$

(x_0, y_0, z_0)

$\vec{n}\langle a, b, c\rangle$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$ scalar equation

$$\boxed{2(x-1) + 3(y-2) + 4(z-1) = 0}$$

$$2x - 2 + 3y - 6 + 4z - 4 = 0$$

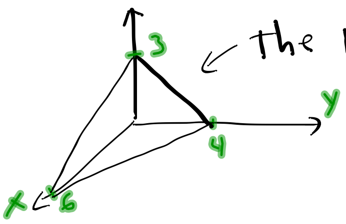
$$\boxed{2x + 3y + 4z = 12}$$ linear equation

Intercepts

x-intercept $\quad y = z = 0 \quad 2x = 12 \Rightarrow x = 6 \Rightarrow (6, 0, 0)$

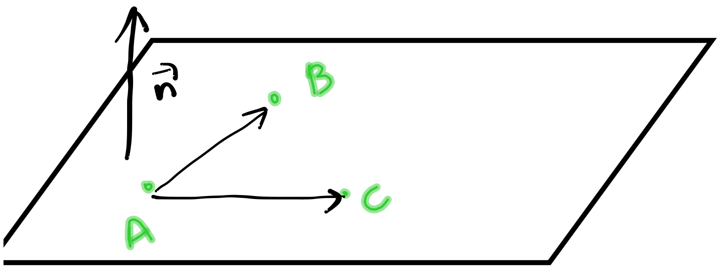
y-intercept $\quad x = z = 0 \quad 3y = 12 \Rightarrow y = 4 \Rightarrow (0, 4, 0)$

z-intercept $\quad x = y = 0 \quad 4z = 12 \Rightarrow z = 3 \Rightarrow (0, 0, 3)$



the portion of the plane in the first octant

EXAMPLE 8. *Determine the equation of the plane through the points* $\underbrace{A(1,1,1)}_{x_0, y_0, z_0}$, $B(0,1,0)$ *and* $C(1,2,3)$.



To find a normal $\vec{n}$

$\vec{n} \perp \vec{AB} = \langle -1, 0, -1 \rangle$

$\vec{n} \perp \vec{AC} = \langle 0, 1, 2 \rangle$ $\Rightarrow$ $\vec{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 0 & -1 \\ 0 & 1 & 2 \end{vmatrix}$

$$\vec{n} = (0-(-1))\hat{i} - (-2-0)\hat{j} + (-1-0)\hat{k}$$

$$= \langle \overset{a}{1}, \overset{b}{2}, \overset{c}{-1} \rangle$$

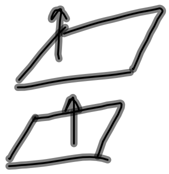
$$1(x-1) + 2(y-1) + (-1)(z-1) = 0$$

or

$$x - 1 + 2y - 2 - z + 1 = 0$$

$$\boxed{x + 2y - z = 2}$$

Two planes are **parallel** if their normal vectors are parallel.



Two planes are **orthogonal** if their normal vectors are orthogonal.

If two planes are not parallel, then they intersect in a straight line and the **angle** between the two planes is defined as the *acute* angle between their normal vectors.

EXAMPLE 9. *Given four planes:*

$$
\begin{array}{lrcrcrcrcl}
P_1: & 2x & + & 3y & + & z & + & 11 & = & 0 \\
P_2: & -4x & - & 6y & - & 2z & + & 77 & = & 0 \\
P_3: & 2x & & & - & 4z & + & 33 & = & 0 \\
P_4: & -2x & + & 3y & + & z & + & 11 & = & 0.
\end{array}
$$

$\vec{n}_1\langle 2,3,1\rangle$
$\vec{n}_2\langle -4,-6,-2\rangle$
$\vec{n}_3\langle 2,0,-4\rangle$
$n_4\langle -2,3,1\rangle$

Determine whether the given pairs of the planes are parallel, orthogonal, or neither. Find the angle between the planes.

(a) P_1 *and* P_2

$\vec{n}_2 = -2\vec{n}_1 \Rightarrow \vec{n}_1 \parallel \vec{n}_2 \Rightarrow P_1 \parallel P_2$

or

$$\cos\theta = \frac{\vec{n}_1\cdot\vec{n}_2}{|n_1|\cdot|n_2|} = \frac{-8-18-2}{\sqrt{4+9+1}\,\sqrt{4+9+1}} = \frac{-28}{2\cdot 14} = -1$$

$\Rightarrow \theta = \pi \Rightarrow \vec{n}_1 \parallel \vec{n}_2$

(b) P_1 *and* P_3

$\vec{n}_1\cdot\vec{n}_3 = \langle 2,3,1\rangle\cdot\langle 2,0,-4\rangle = 4+0-4 = 0 \Rightarrow \vec{n}_1 \perp \vec{n}_3$

$\Rightarrow P_1 \perp P_3$

(c) P_2 *and* P_3

$$\left.\begin{array}{l} P_1 \parallel P_2 \\ P_1 \perp P_3 \end{array}\right\} \Rightarrow P_2 \perp P_3$$

(d) P_1 *and* P_4

$\vec{n}_1\cdot\vec{n}_4 = \langle 2,3,1\rangle\cdot\langle -2,3,1\rangle = -4+9+1 = 6$

$|\vec{n}_1| = \sqrt{2^2+3^2+1^2} = \sqrt{14}$

$|\vec{n}_4| = \sqrt{(-2)^2+3^2+1^2} = \sqrt{14}$

$$\cos\theta = \frac{\vec{n}_1\cdot\vec{n}_4}{|n_1|\cdot|n_4|} = \frac{6}{\sqrt{14}\sqrt{14}} = \frac{6}{14} = \frac{3}{7}$$

$$\theta = \arccos\frac{3}{7} = \ldots$$

EXAMPLE 10. *Find an equation of the line given as intersection of two planes:*

$$\begin{cases} x - y + 3z = 0 & P_1 \\ x + y + 4z = 2 & P_2 \end{cases}$$

Find a point P on the line $L = P_1 \cap P_2$

Set $z=0 \Rightarrow \begin{cases} x-y=0 \\ x+y=2 \end{cases} \Rightarrow x=y=1$

$P(1,1,0)$

Find a direction vector $\vec{v}$ of the line L:

$\left.\begin{array}{l} \vec{v} \perp \vec{n}_1 = \langle 1,-1,3\rangle \\ \vec{v} \perp n_2 = \langle 1,1,4\rangle \end{array}\right\} \Rightarrow \vec{v} = \vec{n}_1 \times \vec{n}_2$

$$\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ 1 & 1 & 4 \end{vmatrix} = (-4-3)\hat{i} - (4-3)\hat{j} + (1-(-1))\hat{k}$$

$$= \langle -7, -1, 2\rangle$$

$$\begin{aligned} x &= 1 \\ y &= 1 \\ z &= 0 \end{aligned} + \begin{matrix} -7t \\ -1t \\ 2t \end{matrix} \Rightarrow \boxed{\begin{aligned} x &= 1-7t \\ y &= 1-t \\ z &= 2t \end{aligned}}$$

$P \qquad \vec{v}$