11.6: Vector Functions and Space Curves

A vector function is a function that takes one or more variables and returns a vector. Let $\mathbf{r}(t)$ be a vector function whose range is a set of 3-dimensional vectors: $\mathbb{R} \to \mathbb{R}^3$

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k},$$

where $x(t), y(t), z(t)$ are functions of one variable and they are called the **component functions**.

A vector function $\mathbf{r}(t)$ is *continuous* if and only if its component functions $x(t), y(t), z(t)$ are continuous.

EXAMPLE 1. *Given*

$$\mathbf{r}(t) = \langle \underbrace{t\ln(t+1)}_{=x(t)}, \underbrace{t^2 \sin t}_{=y(t)}, \underbrace{e^t}_{=z(t)} \rangle .$$

(a) *Find the domain of* $\mathbf{r}(t)$.

$D(x(t)) = \{t+1>0\} = \{t>-1\}$

$D(y(t)) = D(z(t)) = \mathbb{R}$

$D(\vec{r}(t)) = \{t>-1\}$ or $(-1,\infty)$

(b) *Find all* t *where* $\mathbf{r}(t)$ *is continuous.*

$(-1, \infty)$

Space curve is given by **parametric equations***:*

$$C = \{(x, y, z) | x = x(t), y = y(t), z = z(t), \quad t \text{ in } I\},$$

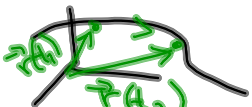
$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

where I is an interval and t is a **parameter***.*

FACT: Any continuous vector-function $\mathbf{r}(t)$ *defines a space curve C that is traced out by the tip of the moving vector* $\mathbf{r}(t)$*.*

Any parametric curve has a **direction of motion** *given by increasing of parameter.*

$t_1 < t_2 \Rightarrow$ Find $\vec{r}(t_1), \vec{r}(t_2)$



EXAMPLE 2. *Describe the curve defined by the vector function (indicate direction of motion):*

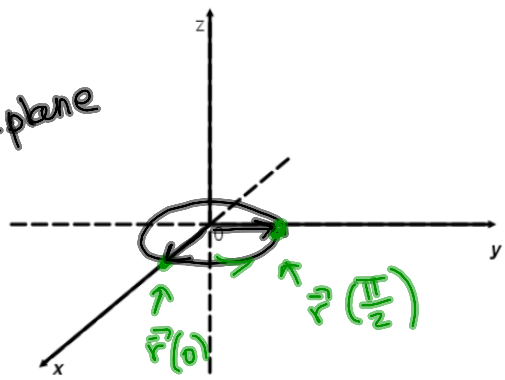
(a) $\mathbf{r}(t) = \langle \cos t, \sin t, 0 \rangle$

$$\left.\begin{array}{l} x = \cos t \\ y = \sin t \\ z = 0 \end{array}\right\} \Rightarrow \begin{array}{l} x^2 + y^2 = 1 \\ z = 0 \end{array}$$

circle in the xy-plane

$\vec{r}(0) = \langle 1, 0, 0 \rangle$

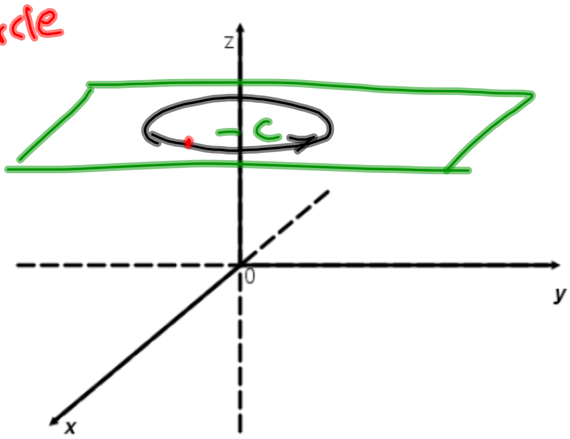
$\vec{r}\left(\frac{\pi}{2}\right) = \langle 0, 1, 0 \rangle$



(b) $\mathbf{r}(t) = \langle \cos at, \sin at, c \rangle$ *where a and c are positive constants.*

$$\left.\begin{aligned} x &= \cos at \\ y &= \sin at \\ z &= c \end{aligned}\right\} \Rightarrow \begin{aligned} x^2 + y^2 &= 1 \\ z &= c \end{aligned}$$

unit circle in the plane $z=c$

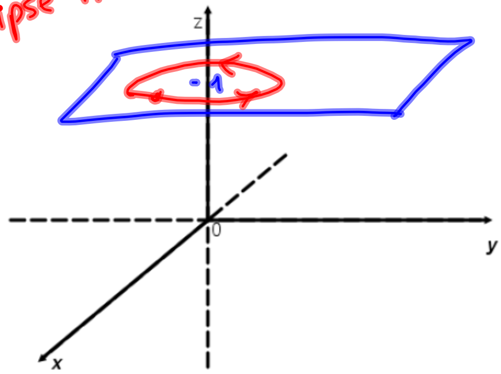


parameter domain

(c) $\mathbf{r}(t) = \langle 2\cos t, 3\sin t, 1 \rangle$, $\underbrace{0 \le t \le 2\pi}_{}$

$$\left.\begin{aligned} x &= 2\cos t \\ y &= 3\sin t \\ z &= 1 \end{aligned}\right\} \Rightarrow \begin{aligned} \left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 &= 1 \\ z &= 1 \end{aligned}$$

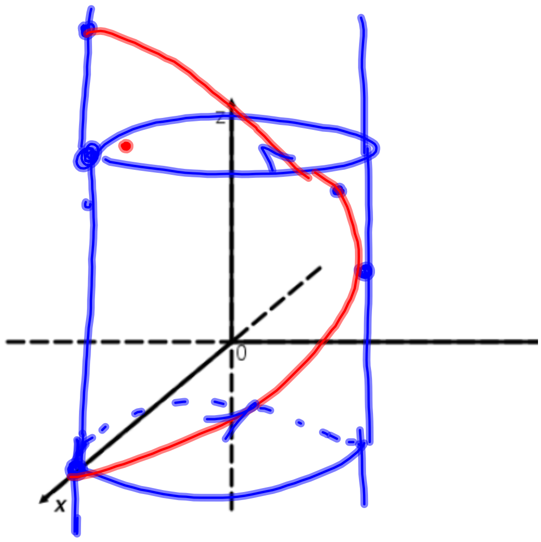
ellipse in the plane $\cancel{x}\, z=1$



(d) $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$

$$\left.\begin{aligned} x &= \cos t \\ y &= \sin t \end{aligned}\right\} \Rightarrow x^2 + y^2 = 1$$

$$z = t$$

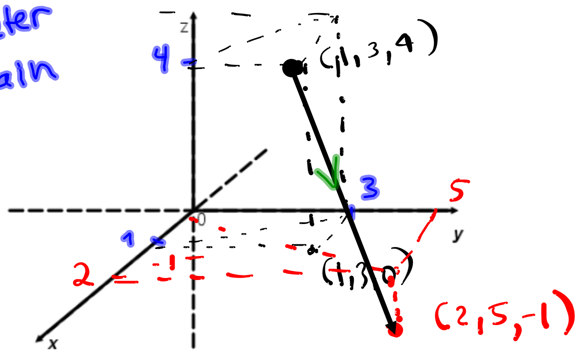


(e) $\mathbf{r}(t) = \langle 1 + t, 3 + 2t, 4 - 5t \rangle$, $\underbrace{0 \le t \le 1}_{\text{parameter domain}}$.

line segment

$\vec{r}(0) = \langle 1, 3, 4 \rangle$

$\vec{r}(1) = \langle 2, 5, -1 \rangle$



EXAMPLE 3. *Show that the the curve given by*

$$\mathbf{r}(t) = \left\langle \sin t, 2\cos t, \sqrt{3}\sin t \right\rangle$$

lies on both a plane and a sphere. Then conclude that its graph is a circle and find its radius.

$$\left.\begin{aligned} x &= \sin t \\ y &= 2\cos t \\ z &= \sqrt{3}\sin t \end{aligned}\right\} \Rightarrow z = \sqrt{3}\underbrace{\sin t}_{x} = x \Rightarrow \boxed{z = \sqrt{3}x}$$

plane through the origin

$$x^2 + y^2 + z^2 = (\sin t)^2 + (2\cos t)^2 + (\sqrt{3}\sin t)^2$$

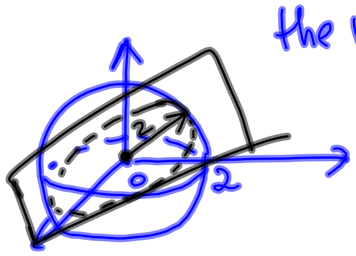
$$= \sin^2 t + 4\cos^2 t + 3\sin^2 t$$

$$= 4\sin^2 t + 4\cos^2 t = 4$$

$4(\sin^2 t + \cos^2 t) = 4 \cdot 1$

$x^2 + y^2 + z^2 = 4$ sphere centered at origin with radius 2

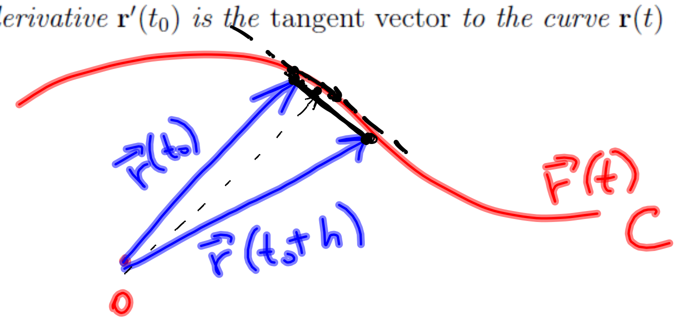
the radius of this circle of intersection is 2



Derivatives: *The derivative* $\mathbf{r}'$ *of a vector function* $\mathbf{r}$ *is defined just as for a real-valued function:*

$$\frac{\mathrm{d}\mathbf{r}(t_0)}{\mathrm{d}t} = \mathbf{r}'(t_0) = \lim_{h\to 0}\frac{\mathbf{r}(t_0+h)-\mathbf{r}(t_0)}{h}$$

if the limit exists. The derivative $\mathbf{r}'(t_0)$ *is the* tangent vector *to the curve* $\mathbf{r}(t)$ *at the point* $\mathbf{r}(t_0) = \langle x(t_0), y(t_0), z(t_0)\rangle$.



THEOREM 4. *If the functions* $x(t), y(t), z(t)$ *are differentiable, then*

$$\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}.$$

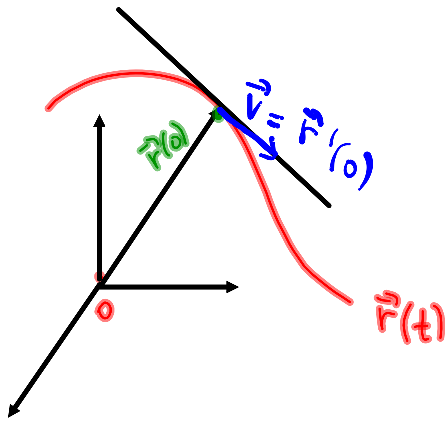
EXAMPLE 5. *Given* $\mathbf{r}(t) = (1+t)^2\mathbf{i} + e^t\mathbf{j} + \sin 3t\mathbf{k}$.

(a) *Find* $\mathbf{r}'(t)$

$$\vec{r}\,'(t) = \left\langle \frac{d}{dt}(1+t)^2, \frac{d}{dt}(e^t), \frac{d}{dt}(\sin 3t)\right\rangle = \langle 2(1+t), e^t, 3\cos 3t\rangle$$

(b) *Find a tangent vector to the curve at* $t = 0$.

$$\vec{v} = \vec{r}\,'(0) = \langle 2, 1, 3\rangle$$



(c) *Find a tangent line to the curve at* $t = 0$.

$$\begin{aligned} x &= x_0 + a\,t \\ y &= y_0 + b\,t \\ z &= z_0 + c\,t \end{aligned}$$

$\vec{r}(0)$ $\quad$ $\vec{v} = \vec{r}\,'(0)$

$\vec{r}(0) = \langle 1, 1, 0\rangle$

$$\begin{aligned} x &= 1 + 2t \\ y &= 1 + 1t \\ z &= 0 + 3t \end{aligned} \Rightarrow \boxed{\begin{aligned} x &= 1+2t \\ y &= 1+t \\ z &= 3t \end{aligned}}$$

(c′) *Find a tangent line to the curve at the point* $< 1, 1, 0 >$.