12.1: Functions of Several Variables

Consider the following formulas:

$$z = 2 - x - 4y \tag{1}$$

$$z = x^2 + y^2 \tag{2}$$

$$z = \sqrt{x^2 + y^2} \tag{3}$$

$$z = \sqrt{1 - x^2 - y^2} \tag{4}$$

$z : \mathbb{R}^2 \to \mathbb{R}$

ordered pair of independent variables (x, y) (input) $\mathbb{R}^2$	$\longrightarrow$	$z = f(x,y)$	$\longrightarrow$	dependent variable z (output) $\mathbb{R}$

DEFINITION 1. *Let $D \subset \mathbb{R}^2$. A* **function** *f* **of two variables** *is a rule that assigns to each ordered pair (x, y) in D a unique real number denoted by $f(x, y)$.*

The set D is the **domain** *of f and the* **range** *of f is the set of values that f takes on, that is $\{f(x, y) | (x, y) \in D\}$.*

REMARK 2. Obviously, one can choose the independent variables arbitrary, for example, $x = f(y, z)$.

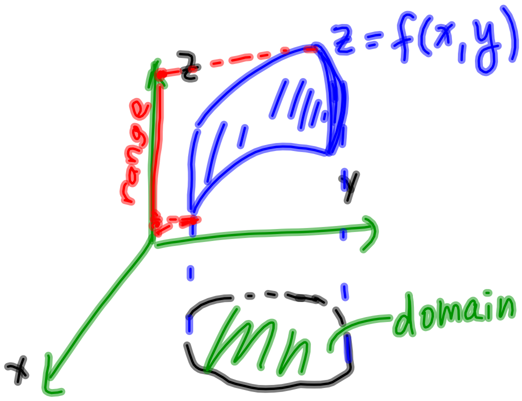
- **GRAPH of $f(x, y)$.**

Recall that a graph of a function f of one variable is a curve C with equation $y = f(x)$.

DEFINITION 3. *The* **graph** *of f with domain D is the set:*

$$S = \left\{(x, y, z) \in \mathbb{R}^3 | z = f(x, y). \ \ (x, y) \in D\right\}.$$

The graph of a function f of two variables is a surface S in three dimensional space with equation $z = f(x, y)$.



EXAMPLE 4. *Find the domain and sketch the graph of the functions (1)-(4). What is the range?*

(1) $z = 2 - x - 4y$ plane

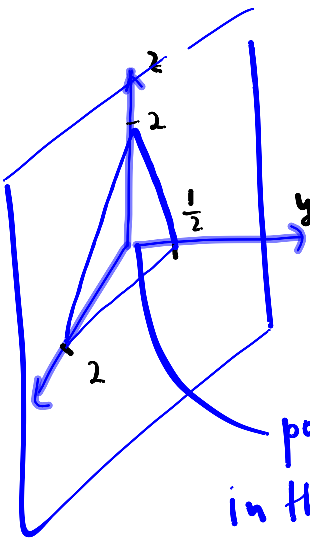
$D = \mathbb{R}^2$

intercepts

$(0,0,2)$

$(0,\frac{1}{2},0)$

$(2,0,0)$



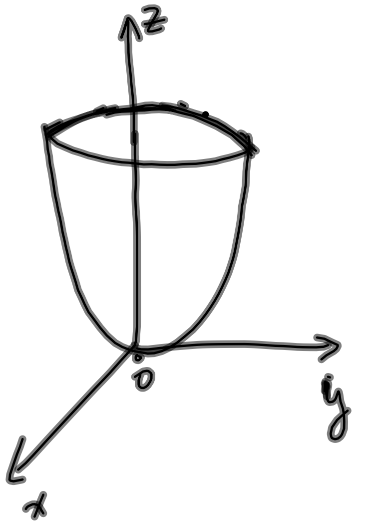
portion of the plane in the 1st octant

<u>Range</u> : $\mathbb{R}$ or $\{z \mid -\infty < z < \infty\}$

(2) $z = x^2 + y^2$

$D = \mathbb{R}^2$

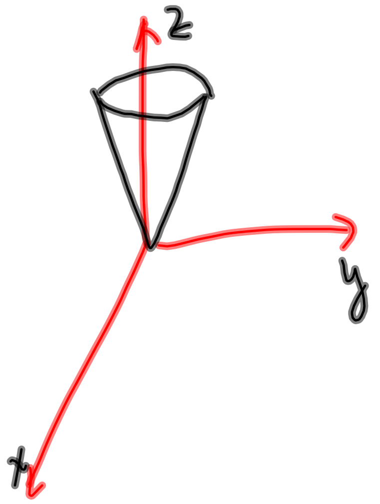
Range $\{z \mid z \geq 0\} = \mathbb{R}_+$



(3) $z = \sqrt{x^2 + y^2}$

$D = \mathbb{R}^2$

Range $\mathbb{R}_+ = \{z \mid z \geq 0\}$

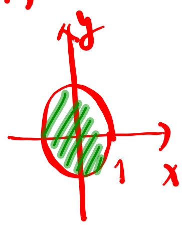


(4) $z = \sqrt{1 - x^2 - y^2}$

$D = \{(x,y) : x^2 + y^2 \leq 1\}$ unit disk

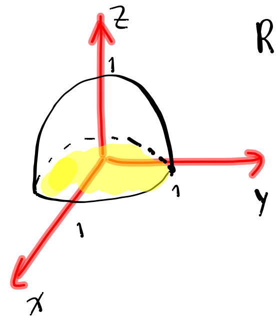
$1 - x^2 - y^2 \geq 0$

$1 \geq x^2 + y^2$



$z^2 = 1 - x^2 - y^2, \; z \geq 0$

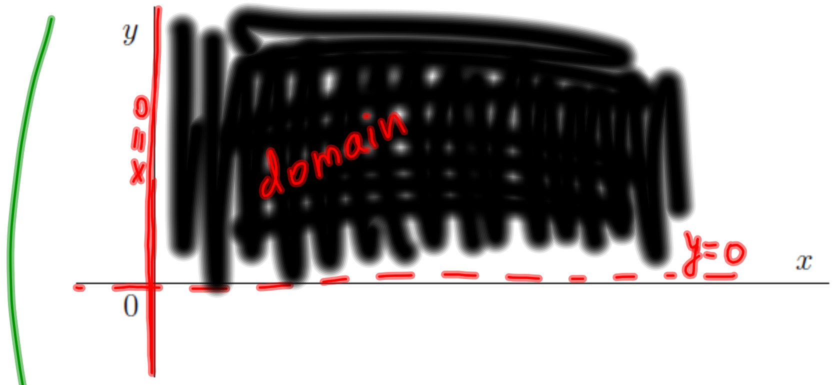
$x^2 + y^2 + z^2 = 1, \; z \geq 0$



Range: $\{z \mid 0 \leq z \leq 1\}$

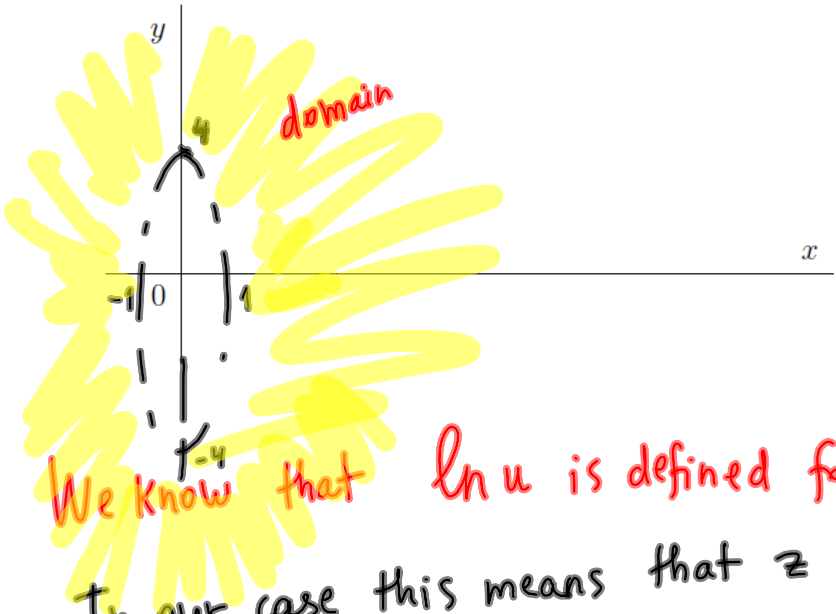
EXAMPLE 5. *Sketch the domain of each of the following:*

(a) $z = \sqrt{x} - \dfrac{5}{\sqrt{y}}$



z is defined for all x, y such that $\left\{ \begin{array}{l} x \geq 0 \\ y > 0 \end{array} \right\}$

(b) $z = \ln(\underbrace{x^2 + \frac{y^2}{16} - 1}_{u})$



We know that $\ln u$ is defined for all $u > 0$

In our case this means that z is defined when

$$x^2 + \frac{y^2}{16} - 1 > 0$$

$$\left\{ x^2 + \frac{y^2}{16} > 1 \right\} = \text{domain}$$

• **LEVEL (CONTOUR) CURVES** method of visualizing functions is the method borrowed from mapmakers. It is a contour map on which points of constant elevation are joined to form level (or contour) curves.

DEFINITION 6. *The* **level (contour) curves** *of a function of two variables are the curves with equations*

$$f(x, y) = k,$$

plane curve (curve in the plane $z=k$)

$Z=K$

where k is a constant in the range of f.

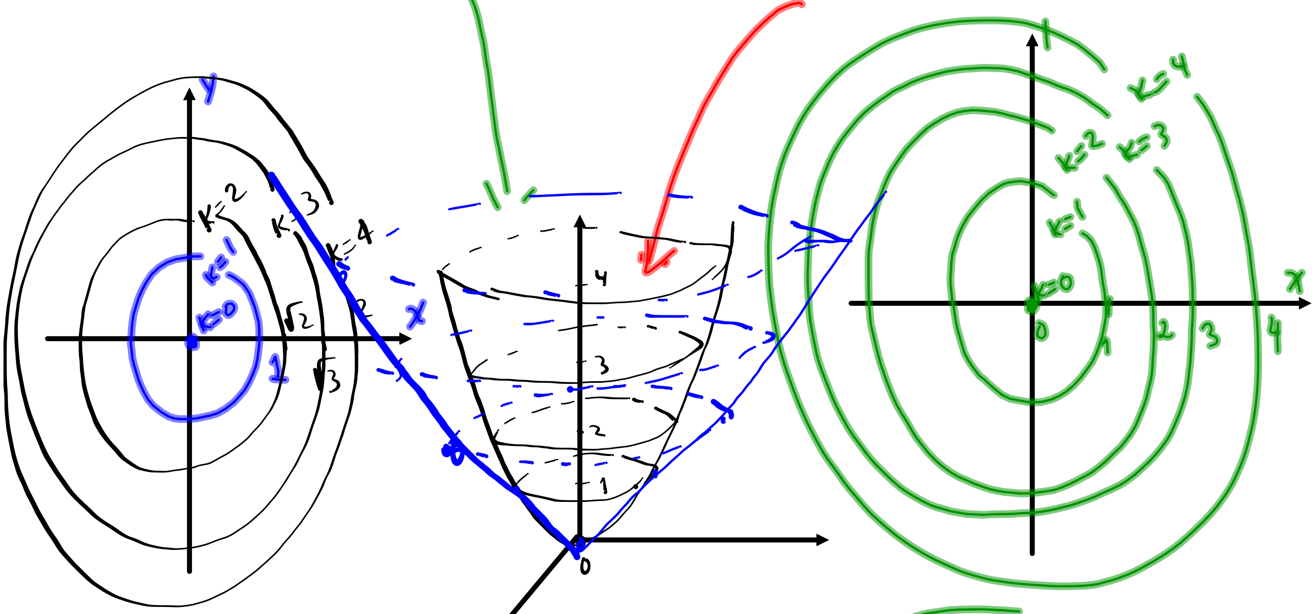
an equiation of a Curve in the xy-plane

A level curve is the locus of all points at which f takes a given value k (it shows where the graph of f has height k).

Sketch contour map.

EXAMPLE 7. *Sketch the level curves of the functions (2) and (3) for the values* $k = 0, 1, 2, 3, 4$:

(2) $z = x^2 + y^2$ (3) $z = \sqrt{x^2 + y^2}$



Level Curves

$x^2+y^2=k$

Note $k \geq 0$

$\sqrt{x^2+y^2} = k$

$x^2+y^2 = k^2 \quad (k \geq 0)$

• **Functions of three variables.** $\mathbb{R}^3 \to \mathbb{R}$

DEFINITION 8. *Let $D \subset \mathbb{R}^3$. A* **function** f **of three variables** *is a rule that assigns to each ordered pair (x, y, z) in D a unique real number denoted by $f(x, y, z)$.*

Examples of functions of 3 variables:

$$f(x, y, z) = x^2 + y^2 + z^2,$$

$$u = xyz$$

$$T(s_1, s_2, s_3) = \ln s_1 + 12s_2 - s_3^{-5}.$$

Emphasize that domains of functions of three variables are regions in three dimensional space.

EXAMPLE 9. *Find the domain of the following function:*

$$f(x, y, z) = \frac{\ln(36 - x^2 - y^2 - z^2)}{\sqrt{x^2 + y^2 + z^2 - 25}}.$$

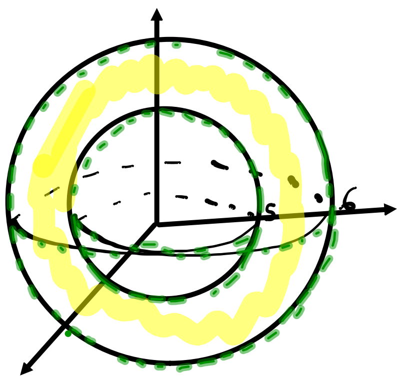
$D(\ln u) = \{u > 0\}$

$\frac{1}{\sqrt{v}} = \{v > 0\}$

$$\begin{cases} 36 - x^2 - y^2 - z^2 > 0 \\ x^2 + y^2 + z^2 - 25 > 0 \end{cases}$$

$$\begin{cases} 36 > x^2 + y^2 + z^2 \\ x^2 + y^2 + z^2 > 25 \end{cases}$$

$D(f) = \{(x, y, z) \text{ in } \mathbb{R}^3 \mid 25 < x^2 + y^2 + z^2 < 36\}$



Note that for functions of three variables it is impossible to visualize its graph. However we can examine them by their **level surfaces**:

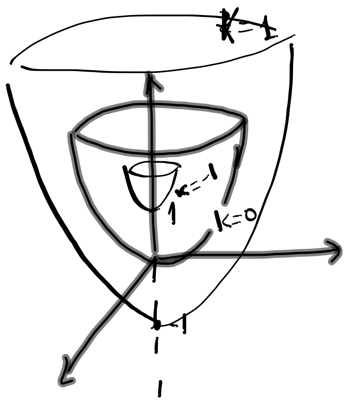
$$f(x, y, z) = k$$

where k is a constant in the range of f. If the point (x, y, z) moves along a level surface, the value of $f(x, y, z)$ remains fixed.

EXAMPLE 10. *Find the level surfaces of the function* $u = x^2 + y^2 - z.$ $= k$

$x^2 + y^2 - z = k$ surface in $\mathbb{R}^3$

$z + k = x^2 + y^2$ paraboloid



REMARK 11. For any function there exist a unique level curve (surface) through given point!!!

$z = f(x, y)$

level curves $\quad f(x, y) = k$

If for example we have

$$\begin{cases} f(1,2) = 3 \\ f(1,2) = 4 \end{cases}$$ impossible