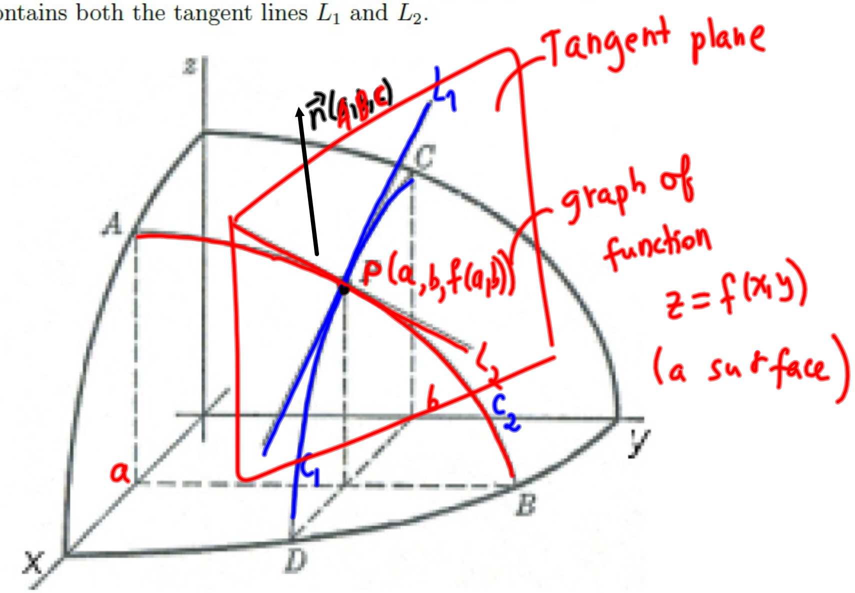
12.4: Tangent Planes to the graph of the function $z=f(x,y)$ and Differentials

Suppose that $f(x, y)$ has continuous first partial derivatives and a surface S has equation $z = f(x, y)$. Let $P(x_0, y_0, z_0)$ be a point on S, i.e. $z_0 = f(x_0, y_0)$.

Denote by C_1 the trace to $f(x, y)$ for the plane $y = y_0$ and denote by C_2 the trace to $f(x, y)$ for the plane $x = x_0$. let L_1 be the tangent line to the trace C_1 and let L_2 be the tangent line to the trace C_2.

The **tangent plane** to the surface S (or to the graph of $f(x, y)$) at the point P is defined to be the plane that contains both the tangent lines L_1 and L_2.



$$\overset{?}{A}(x-a) + \overset{?}{B}(y-b) + \overset{?}{C}(z - f(a,b)) = 0$$

$$\boxed{\vec{n}\,(\underset{A}{f_x(a,b)},\ \underset{B}{f_y(a,b)},\ \underset{C}{-1})}$$

normal vector to the tangent plane at $(a, b, f(a,b))$

$$f_x(a,b)(x-a) + f_y(a,b)(y-b) - (z - f(a,b)) = 0$$

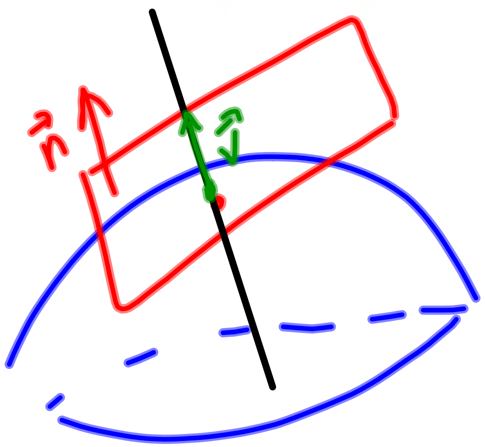
THEOREM 1. *An equation of the tangent plane to the graph of the function $z = f(x, y)$ at th point $P(x_0, y_0, f(x_0, y_0))$ is*

$$z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

CONCLUSION: A normal vector to the tangent plane to the surface $z = f(x, y)$ at the poin $P(x_0, y_0, f(x_0, y_0))$ is

$$\mathbf{n} = \mathbf{n}(x_0, y_0) = \langle f_x(x_0, y_0), f_y(x_0, y_0), -1 \rangle.$$

The line through the point $P(x_0, y_0, f(x_0, y_0))$ parallel to the vector $\mathbf{n}$ is perpendicular to th above tangent plane. This line is called **the normal line** to the surface $z = f(x, y)$ at P. I follows that this normal line can be expressed parametrically as



$\vec{v} \parallel \vec{n}$

EXAMPLE 2. *Find an equation of the tangent plane to the graph of the function* $z = x^2 + y^2 + 8$ *at the point* $(1,1)$.

$\overset{\shortparallel}{z(x,y)}$

x_0, y_0

$z(1,1) = 1^2 + 1^2 + 8 = 10$

Tangent point $P(1,1,10)$

Normal vector $\vec{N} = \langle \underset{2}{z_x(1,1)}, \underset{2}{z_y(1,1)}, -1 \rangle = \langle 2, 2, -1 \rangle$

$z_x = 2x$

$z_y = 2y$

$2(x-1) + 2(y-1) - (z-10) = 0$

$$\boxed{2x + 2y - z + 6 = 0}$$

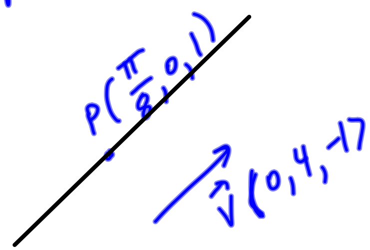
Ex.3 Find param. equations for the normal line to $z = e^{4y}\sin(4x)$ at the point $P\left(\frac{\pi}{8}, 0, 1\right)$.

$$\vec{n} = \left\langle z_x\left(\frac{\pi}{8}, 0\right),\ z_y\left(\frac{\pi}{8}, 0\right), -1 \right\rangle$$

$$z_x = 4e^{4y}\cos(4x) \qquad z_x\left(\frac{\pi}{8}, 0\right) = 0$$

$$z_y = 4e^{4y}\sin(4x) \qquad z_y\left(\frac{\pi}{8}, 0\right) = 4$$

a normal vector for tangent plane $\rightarrow \vec{n} = \langle 0, 4, -1 \rangle = \vec{v}$ a direction vector for normal line



$$x = \frac{\pi}{8} + t \cdot 0$$
$$y = 0 + t \cdot 4$$
$$z = 1 + t \cdot (-1)$$

$$\boxed{\begin{aligned} x &= \frac{\pi}{8} \\ y &= 4t \\ z &= 1 - t \end{aligned}}$$

$$df(x) = f'(x)\,dx$$

$$\Delta x = dx \quad , \quad \Delta f \approx df$$

The **differentials** $\mathrm{d}x$ and $\mathrm{d}y$ are independent variables. The **differential** $\mathrm{d}z$ (or the **total differential**) is defined by

$$\mathrm{d}z = \frac{\partial z}{\partial x}\mathrm{d}x + \frac{\partial z}{\partial y}\mathrm{d}y.$$

FACT: $\Delta z \approx \mathrm{d}z$. $= f(a+\Delta x, b+\Delta y) - f(a,b)$

$x = a + \Delta x$

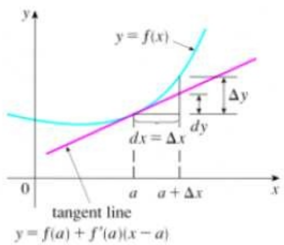
$y = b + \Delta y$

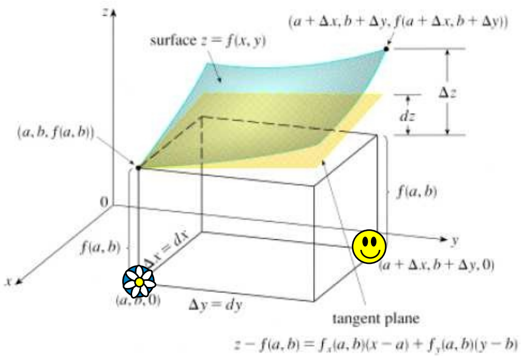
This implies:

$$\boxed{f(a+\Delta x, b+\Delta y) \approx f(a,b) + \mathrm{d}z(a,b)}$$

or

$$f(x,y) \approx f(a,b) + dz(a,b)$$





[1]

[1] the pictures are from our textbook

EXAMPLE 4. *Use differentials to find an approximate value for* $\sqrt{1.03^2 + 1.98^3}$.

$$\sqrt{1.03^2 + 1.98^3} = f(1.03, 1.98)$$

$$f(x,y) = \sqrt{x^2+y^3}$$

Note $f(1,2) = \sqrt{1^2+2^3} = \sqrt{9} = 3$

$$dx = \Delta x = 1.03 - 1 = 0.03$$

$$dy = \Delta y = 1.98 - 2 = -0.02$$

$$f(1.03, 1.98) \approx f(1,2) + df(1,2) = 3 + df(1,2)$$

$$f_x = \frac{\partial}{\partial x}\left(\sqrt{x^2+y^3}\right) = \frac{1}{\cancel{2}\sqrt{x^2+y^3}} \cdot \cancel{2}x$$

$$f_x(1,2) = \frac{1}{3}$$

$$f_y = \frac{\partial}{\partial y}\left(\sqrt{x^2+y^3}\right) = \frac{3y^2}{2\sqrt{x^2+y^3}}$$

$$f_y(1,2) = \frac{3 \cdot 4}{2 \cdot 3} = 2$$

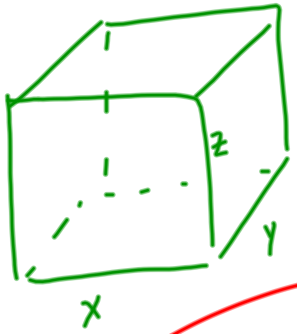
$$\Rightarrow df(1,2) = f_x(1,2)dx + f_y(1,2)dy = \frac{1}{3} \cdot 0.03 + 2 \cdot (-0.02) = 0.01 - 0.04 = -0.03$$

$$\sqrt{1.03^2+1.98^3} = f(1.03, 1.98) \approx 3 + (-0.03) = \boxed{2.97}$$

If $u = f(x, y, z)$ then the differential $\mathrm{d}u$ at the point $(x, y, z) = (a, b, c)$ is defined in terms of the differentials $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}z$ of the independent variables:

$$\mathrm{d}u(a, b, c) = f_x(a, b, c)\mathrm{d}x + f_y(a, b, c)\mathrm{d}y + f_z(a, b, c)\mathrm{d}z.$$

EXAMPLE 5. *The dimensions of a closed rectangular box are measured as 80 cm, 60 cm and 50 cm, respectively, with a possible error of* 0.2 *cm in each dimension. Use differentials to estimate the maximum error in calculating the surface area of the box.*



$$S = 2(xy + xz + yz)$$
$$\parallel$$
$$S(x, y, z)$$

$$\Delta S(80, 60, 50) \approx \underline{dS(80, 60, 50)}$$

where $\Delta x = \Delta y = \Delta z = 0.2$
$\ \ \parallel \quad\ \parallel \quad\ \parallel$
$\ \ dx \quad dy \quad dz$

$$\text{error} \approx S_x(80, 60, 50)\,dx + S_y(80, 60, 50)\,dy + S_z(80, 60, 50)\,dz$$

$$S_x = \frac{\partial}{\partial x}\left(2(xy + xz + yz)\right) = 2(y + z) = 2(60 + 50) = 220$$

$$S_y = 2(x + z) = 2(80 + 50) = 260$$

$$S_z = 2(x + y) = 2(80 + 60) = 280$$

$$\text{error} \approx 0.2(220 + 260 + 280) = 0.2 \cdot 760 = 152 \text{ cm}^2$$

A function $f(x, y)$ is **differentiable** *at (a, b) if its partial derivatives f_x and f_y exist and are continuous at (a, b).*

For example, all polynomial and rational functions are differentiable on their natural domains.

Let a surface S be a graph of a differentiable function f. As we zoom in toward a point on the surface S , the surface looks more and more like a plane (its tangent plane) and we can approximate the function f by a linear function of two variables.

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) =: L(x, y).$$