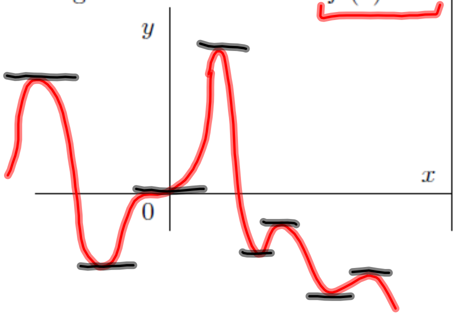
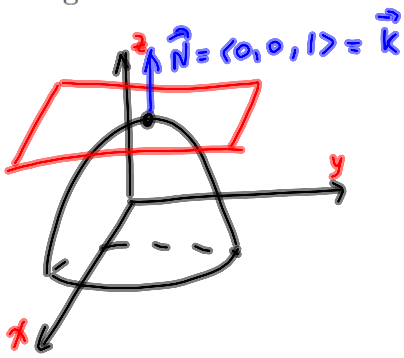
12.7: Maximum and minimum values

Function $y = f(x)$ Calc I	Function of two variables $z = f(x, y)$ Calc III
DEFINITION 1. *A function $f(x)$ has a local maximum at $x = a$ if $f(a) \geq f(x)$ when x is near a (i.e. in a neighborhood of a). A function f has a local minimum at $x = a$ if $f(a) \leq f(x)$ when x is near a.*	DEFINITION 2. *A function $f(x, y)$ has a local maximum at $(x, y) = (a, b)$ if $f(a, b) \geq f(x, y)$ when (x, y) is near (a, b) (i.e. in a neighborhood of (a, b)). A function f has a local minimum at $(x, y) = (a, b)$ if $f(a, b) \leq f(x, y)$ when (x, y) is near (a, b).*
If the inequalities in this definition hold for ALL points x in the domain of f, then f has an **absolute max** (or **absolute min**) at a	If the inequalities in this definition hold for ALL points (x, y) in the domain of f, then f has an **absolute maximum** (or **absolute minimum**) at (a, b).

If the graph of f has a tangent line at a local extremum, then the tangent line is horizontal: $f'(a) = 0$.



If the graph of f has a tangent plane at a local extremum, then the tangent PLANE is horizontal.



THEOREM 3. *If f has a local extremum (that is, a local maximum or minimum) at (a, b) and first-order partial derivatives exist there, then*

$$f_x(a, b) = f_y(a, b) = 0 \quad (\textit{or, equivalently}, \nabla f(a, b) = 0.)$$

Normal to tangent plane of $z = f(x, y)$

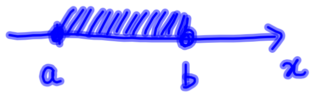
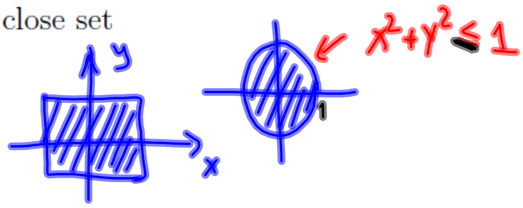
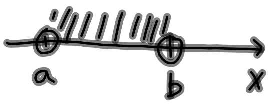
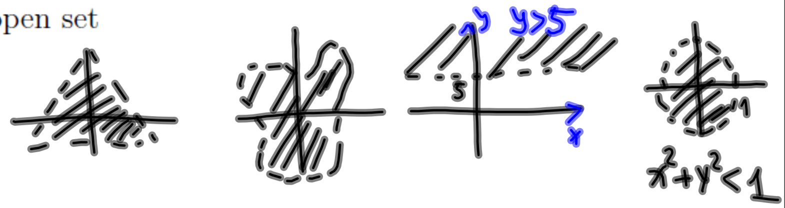
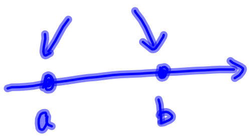
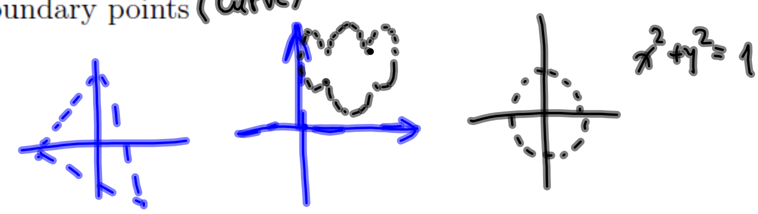
$$\vec{N} = \langle f_x, f_y, -1 \rangle \parallel \vec{k} = \langle 0, 0, 1 \rangle \Rightarrow \begin{array}{l} f_x = 0 \\ f_y = 0 \end{array}$$

DEFINITION 4. *A point (a, b) such that $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or one of this partial derivatives does not exist, is called a* **critical point** *of f.*

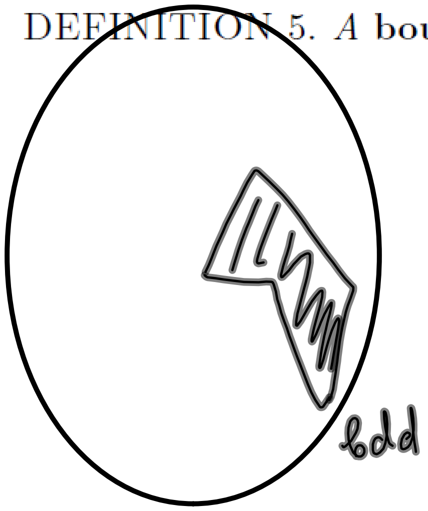
At a critical point, a function could have a local max or a local min, or neither.
We will be concerned with two important questions:

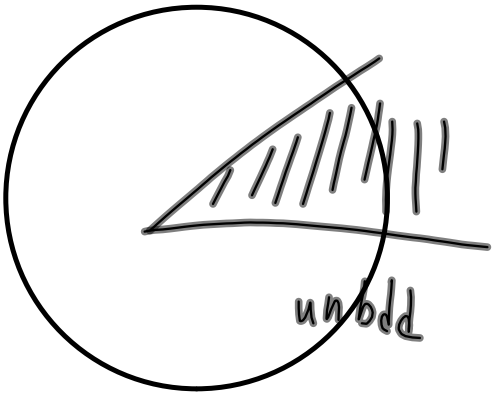
- Are there any local or absolute extrema?
- If so, where are they located?

(Regions, domains) SETS in $\mathbb{R}^2$

in $\mathbb{R}$	in $\mathbb{R}^2$
close interval $[a, b]$ 	close set 
open interval (a, b) 	open set 
end points of an interval 	boundary points (curve) 

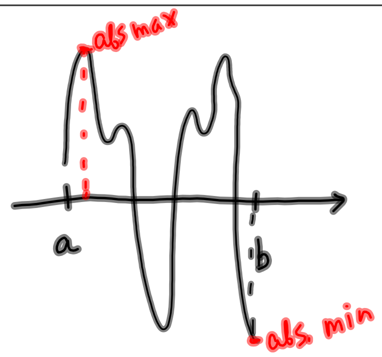
DEFINITION 5. *A* **bounded set** *in* $\mathbb{R}^2$ *is one that contained in some disk.*

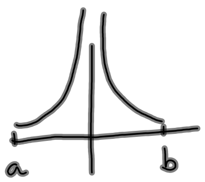


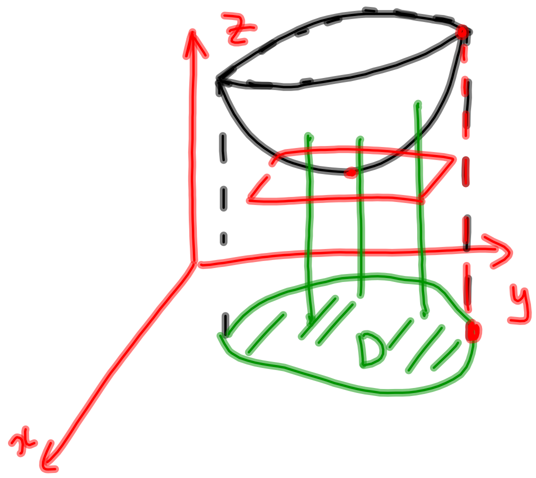


THE EXTREME VALUE THEOREM:

Function $y = f(x)$	Function of two variables $z = f(x, y)$
If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(x_1)$ and an absolute minimum value $f(x_2)$ at some points x_1 and x_2 in $[a, b]$.	If f is continuous on a closed bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.





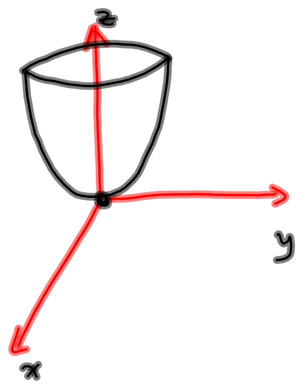


EXAMPLE 6. *Find extreme values of* $f(x, y) = x^2 + y^2$.

	Local	Absolute
Maximum	NO	NO
Minimum	(0,0)	(0,0)

Domain: $\mathbb{R}^2$ (unclosed unbdd)

$z = x^2 + y^2$



Note $\nabla f = \langle 2x, 2y \rangle = \langle 0, 0 \rangle$

$\Downarrow$

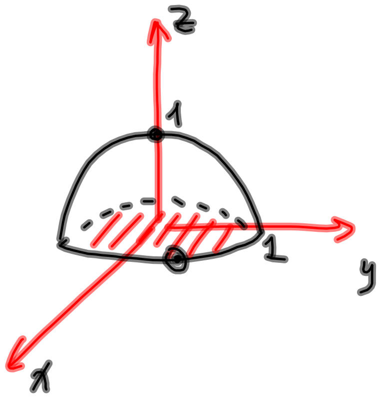
$(x, y) = (0, 0)$ is critical point

EXAMPLE 7. *Find extreme values of* $f(x,y)=\sqrt{1-x^2-y^2}$.

	Local	Absolute
Maximum	at $(0,0)$	at $(0,0)$ and $f(0,0)=1$ ← abs. max value.
Minimum	NO	on circle $x^2+y^2=1$ and abs. min value is 0.

$z=\sqrt{1-x^2-y^2}$ half-sphere

$(z^2=1-x^2-y^2,\ z\geq 0)$



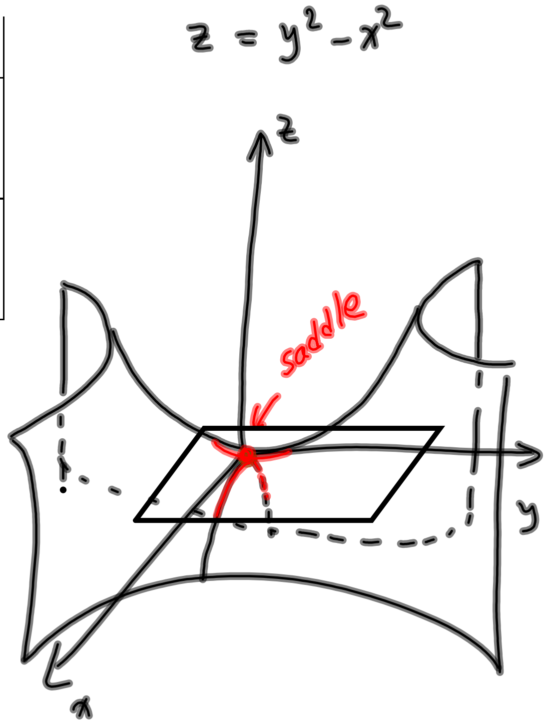
Domain: $x^2+y^2\leq 1$ unit disk

closed & bdd

EXAMPLE 8. *Find extreme values of* $f(x, y) = y^2 - x^2$.

	Local	Absolute
Maximum	NO	NO
Minimum	NO	NO

Domain: $\mathbb{R}^2$ unbdd & unclosed



Crit. points

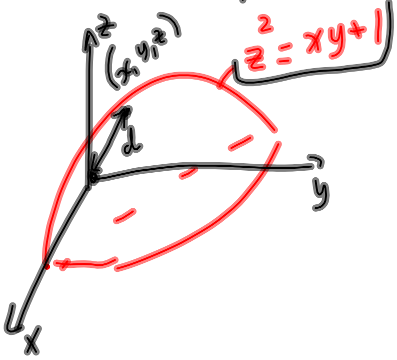
$f_x = 0 \Rightarrow -2x = 0 \Rightarrow x = 0$

$f_y = 0 \Rightarrow 2y = 0 \Rightarrow y = 0$

$\Rightarrow (0,0)$

REMARK 9. Example 8 illustrates so called **saddle point** of f. Note that the graph of f crosses its tangent plane at (a, b).

EXAMPLE 10. *Find the points on the surface $z^2 = xy + 1$ that are closest to the origin.*



$$\begin{cases} d = \sqrt{x^2+y^2+z^2} \to \min \\ \text{subject to } z^2 = xy+1 \end{cases}$$

Equivalent to

$$d^2 = \boxed{x^2+y^2+xy+1 = f(x,y) \to \min}$$
for all x, y

Note: domain is unclosed and unbdd
$D = \mathbb{R}^2$

Critical points

$$\begin{matrix} f_x = 0 \\ f_y = 0 \end{matrix} \Rightarrow \begin{matrix} f_x = 2x + y = 0 \Rightarrow y = -2x & \Rightarrow y = 0 \\ f_y = 2y + x = 0 \quad -4x + x = 0 & \Rightarrow x = 0 \end{matrix}$$

Only one critical point $(x, y) = (0,0)$

Next step: Determine whether that critical point is local max or local min. ~~postpone~~

We can use Second Derivative test to determine if $(0,0)$ is the lowest in its neighborhood:

$$f_{xx} = 2 > 0, \quad f_{yy} = 2, \quad f_{xy} = 1 \quad \Rightarrow D(0,0) = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 > 0$$

$(0,0)$ is local min

According to the nature of the problem this point must be point of abs. minimum.

$$z^2 = xy + 1 \quad \Rightarrow \quad z = \pm\sqrt{xy+1}$$
$$z(0,0) = \pm 1$$

Answer: There are two such points $(x,y,z) = \boxed{(0,0,\pm 1)}$

ABSOLUTE MAXIMUM AND MINIMUM VALUES on a closed bounded set.

THE EXTREME VALUE THEOREM (see above)

Calc I

To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical points of f in (a, b).

2. Find the extreme values of f at the endpoints of the interval.

3. The largest of the values from steps 1&2 is the absolute max value; the smallest of the values from steps 1&2 is the absolute min value.

Calc II

To find the absolute max and min values of a continuous function f on a $\boxed{\textit{closed bounded set } D}$:

1. Find the values of f at the critical points of f in D.

2. Find the extreme values of f on the boundary of D. (This usually involves the Calculus I approach for this work.)

3. The largest of the values from steps 1&2 is the absolute maximum value; the smallest of the values from steps 1&2 is the absolute minimum value.

SORT ALL points from steps 1&2

- The quantity to me maximized/minimized is expressed in terms of variables (as few as possible!)
- Any constraints that are presented in the problem are used to reduce the number of variables to the point they are independent,
- After computing partial derivatives and setting them equal to zero you get purely algebraic problem (but it may be hard.)
- Sort out extreme values to answer the original question.

Daily Grade #2

Name ______________________

Find the absolute maximum and absolute minimum vales of the function

$$f(x)=2x^3+3x^2-12x$$

on the interval [0,2].

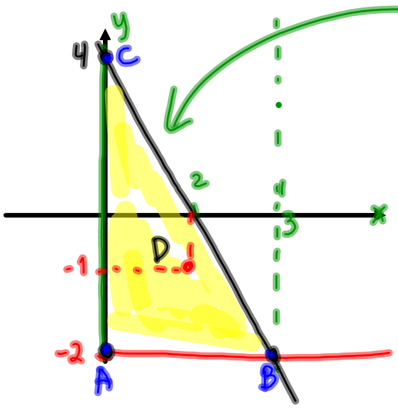
closed and bdd

EXAMPLE 11. *A lamina occupies the region* $D = \{(x,y) : 0 \le x \le 3, \ -2 \le y \le 4-2x\}$. *The temperature at each point of the lamina is given by*

$$\boxed{T(x,y) = 4(x^2 + xy + 2y^2 - 3x + 2y) + 10.}$$

Find the hottest and coldest points of the lamina.

Equivalently, find absolute extrema of $T(x,y)$ on D.



STEP 1 Find critical points in D

$T_x = 4(2x+y-3) = 0 \Rightarrow y = 3-2x$

$T_y = 4(x+4y+2) = 0$

$x + 4(3-2x) + 2 = 0$

$x + 12 - 8x + 2 = 0$

$7x = 14 \Rightarrow x = 2$

$\rightarrow y = 3-2\cdot 2 = -1$

Crit. point $(2,-1)$ in D

$T(2,-1) = 4(4 - \cancel{2} + \cancel{2} - 6 - 2) + 10 = -16 + 10 = \boxed{-6}$

STEP 2 The only other places an extreme value can occur is on the boundary of the plate.

Boundary

$\partial D = AB \cup BC \cup CA$

Common points (= corner point = vertices):

$T(A) = T(0,-2) = 4(8-4)+10 = \boxed{26}$

$T(B) = T(3,-2) = \boxed{2}$

$T(C) = T(0,4) = \boxed{170}$

Edges: Reduce number of variables $\left((x,y) \begin{smallmatrix} \nearrow x \\ \text{OR} \\ \searrow y \end{smallmatrix} \right)$

AB: $y = -2$, $0 < x < 3$

$$T\Big|_{AB} = T(x,-2) = 4(x^2 - 2x + 8 - 3x - 4) + 10 = h(x)$$

Find critical points of $h(x)$ on $(0,3)$

$h'(x) = 0 \Leftrightarrow 4(2x - 2 - 3) = 0 \Rightarrow 2x = 5 \Rightarrow x = \frac{5}{2}$

note $0 < \frac{5}{2} < 3$

$T(\frac{5}{2},-2) = h(\frac{5}{2}) = \boxed{1}$

$BC: \quad y = 4 - 2x, \quad 0 < x < 3$

$$T\Big|_{BC} = T(x, 4-2x)$$

$$= 4\big(\underbrace{x^2 + x(4-2x)} + 2(4-2x)^2 - 3x + 2(4-2x)\big) + 10$$

$$= 4\left(x^2 + 4x - 2x^2 + 2(4-2x)^2 - 3x + 8 - 4x\right) + 10$$

$$= 4\left(-x^2 - 3x + 32 - 32x + 8x^2 + 8\right) + 10 = g(x)$$

Find critical points of $g(x)$ on $(0,3)$

$$g'(x) = 4(-2x - 35 + 16x) = 0$$

$$35 = 14x \Rightarrow x = \frac{35}{14} = \frac{5}{2}$$

$$g\left(\tfrac{5}{2}\right) = T\left(\tfrac{5}{2}, -1\right) = \boxed{25}$$

$CA: \quad x = 0 \quad -2 < y < 4$

$$T(0,y) = 4(2y^2 + 2y) + 10 = m(y)$$

$$m'(y) = 0 \iff 4(4y+2) = 0 \Rightarrow y = -\tfrac{1}{2}$$

$$m\left(-\tfrac{1}{2}\right) = T\left(0, -\tfrac{1}{2}\right) = 4\left(2\cdot\tfrac{1}{4} + 2\cdot\left(-\tfrac{1}{2}\right)\right) + 10$$

$$= 4\left(\tfrac{1}{2} - 1\right) + 10 = \boxed{8}$$

STEP 3 SORT OUT

$\max_{D} T(x,y) = T(0,4) = 170 \Rightarrow (0,4)$ is the hottest point

$\min_{D} T(x,y) = T(2,-1) = -6 \Rightarrow$

$(2,-1)$ is the coldest point

Second derivatives test:

Calc I

Suppose f'' is continuous near a and $f'(c) = 0$ (i.e. a is a critical point).

- If $f''(c) > 0$ then $f(c)$ is a local minimum.
- If $f''(c) < 0$ then $f(c)$ is a local maximum.

NOTE:

- If $f''(c) = 0$, then the test gives no information.

Calc III

Suppose that the second partial derivatives of f are continu ous near (a, b) and $\nabla f(a, b) = \mathbf{0}$ (i.e. (a, b) is a critical point

Let $\mathcal{D} = \mathcal{D}(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$

- If $\mathcal{D} > 0$ and $f_{xx}(a, b) > 0$ then $f(a, b)$ is a local minimun
- If $\mathcal{D} > 0$ and $f_{xx}(a, b) < 0$ then $f(a, b)$ is a local maximun
- If $\mathcal{D} < 0$ then $f(a, b)$ is not a local extremum (saddle point

If $\mathcal{D} = 0$ or does not exist, then the test gives no informatio fails.

To remember formula for $\mathcal{D}$:

$$\mathcal{D} = f_{xx} f_{yy} - [f_{xy}]^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

EXAMPLE 12. *Use the Second Derivative Test to confirm that a local cold point of the lamina in the previous Example is* $(2,-1)$.

$$T(x,y) = 4(x^2 + xy + 2y^2 - 3x + 2y) + 10.$$

$T_x = 4(2x+y-3)$

$T_y = 4(x+4y+2)$

(2,-1) is critical point

$T_{xx} = 8 > 0$

$T_{yy} = 16$

$T_{xy} = 4$

$$D = \begin{vmatrix} T_{xx} & T_{xy} \\ T_{xy} & T_{yy} \end{vmatrix} = \begin{vmatrix} 8 & 4 \\ 4 & 16 \end{vmatrix} = 8 \cdot 16 - 16 > 0$$

By 2nd Der. Test

(2,-1) is local min ⇒ a local cold point

EXAMPLE 13. *Find the local extrema of* $f(x,y) = x^3 + y^3 - 3xy$.

Solution: Find critical points:

$f_x = 3x^2 - 3y = 0 \Rightarrow x^2 = y$

$f_y = 3y^2 - 3x = 0 \rightarrow y^2 - x = 0 \Rightarrow x^4 - x = 0$

$x(x^3 - 1) = 0$

$x = 0$, $y = 0^2 = 0$

$\Rightarrow x^3 = 1$, $x = 1$, $y = 1^2 = 1$

Critical points: $(0,0)$ & $(1,1)$

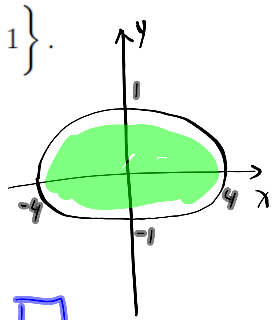
Calculate the second partial derivatives and $\mathcal{D}$.

	$(0,0)$	$(1,1)$
$f_{xx} = 6x$	0	$6 > 0$
$f_{xy} = -3$	-3	-3
$f_{yy} = 6y$	0	6
$\mathcal{D} = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$	$D = \begin{vmatrix} 0 & -3 \\ -3 & 0 \end{vmatrix} = -9 < 0$	$\begin{vmatrix} 6 & -3 \\ -3 & 6 \end{vmatrix} = 36 - 9 > 0$
	saddle	local min

EXAMPLE 14. *The mountain is defined by* $\boxed{z = xy}$ *in the elliptical domain*

$$D = \left\{(x, y) \mid \frac{x^2}{16} + y^2 \le 1\right\}.$$

closed & bdd



(a) *Find the top of the mountain.*

$\parallel$

Find abs. max of z on D

STEP 1 Crit. points in D

$z_x = y$
$z_y = x$
$\Rightarrow (0,0)$ in D $\Rightarrow$ $z(0,0) = \boxed{0}$

STEP 2 Boundary $\underline{\frac{x^2}{16} + y^2 = 1}$

$\nearrow$ Method 1

$y = \pm\sqrt{1 - \frac{x^2}{16}}$

$z = xy = x\sqrt{1 - \frac{x^2}{16}}$

and $z = xy = x(-\sqrt{\quad})$ and so on...

Parameterize the boundary

$x = 4\cos t$, $y = \sin t$, $0 \le t \le 2\pi$

we can restrict the parameter domain to $\underbrace{0 \le t \le \pi}$ $\Leftarrow$

Symmetry

$z(x,y) = xy$

$z(x,y) = z(-x,-y)$

If (x,y) belongs to ellipse then $(\pm x, \pm y)$ also on ellipse

$z\big|_{\text{boundary}} = z(4\cos t, \sin t) = 4\cos t \sin t$

$= 2\sin 2t = h(t)$

Look for critical points on $(0, \pi)$

$h'(t) = 4\cos 2t = 0 \Rightarrow 2t = \frac{\pi}{2} \Rightarrow t = \pi/4$

or $2t = \frac{3\pi}{2} \Rightarrow t = 3\pi/4$

$h\left(\frac{\pi}{4}\right) = z\left(4\cos\frac{\pi}{4}, \sin\frac{\pi}{4}\right) = z\left(2\sqrt{2}, \frac{\sqrt{2}}{2}\right) = 2\sin\left(2\frac{\pi}{4}\right) = \boxed{2}$

$\parallel$ $z\left(-2\sqrt{2}, -\frac{\sqrt{2}}{2}\right)$

$h\left(\frac{3\pi}{4}\right) = z\left(4\cos\frac{3\pi}{4}, \sin\frac{3\pi}{4}\right) = z\left(-2\sqrt{2}, \frac{\sqrt{2}}{2}\right) = \boxed{-2}$

$\parallel$ $z\left(2\sqrt{2}, -\frac{\sqrt{2}}{2}\right)$

By Symmetry

End points $h(0) = z(4\cos 0, \sin 0) = z(4,0) = \boxed{0}$

$h(\pi) = z(4\cos\pi, \sin\pi) = z(-4,0) = \boxed{0}$

The mountain has 2 peak which are

$(x,y,z) =$: $\left(-2\sqrt{2}, -\frac{\sqrt{2}}{2}, 2\right)$ and $\left(2\sqrt{2}, \frac{\sqrt{2}}{2}, 2\right)$

(b) *Is the critical point found in the previous item the highest or the lowest in its neighborhood?*

Determine if at $(0,0)$ we have local max, local min or neither

Use Second derivative Test

$z = xy$

$z_x = y$

$z_y = x$

$z_{xx} = 0$

$z_{yy} = 0$

$z_{xy} = 1$

$$D(0,0) = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 < 0 \Rightarrow$$

$\Rightarrow$ $(0,0)$ saddle point.

Answer: Not highest and not lowest in its neighborhood.

There is a pass at $(0,0)$ (i.e. saddle)