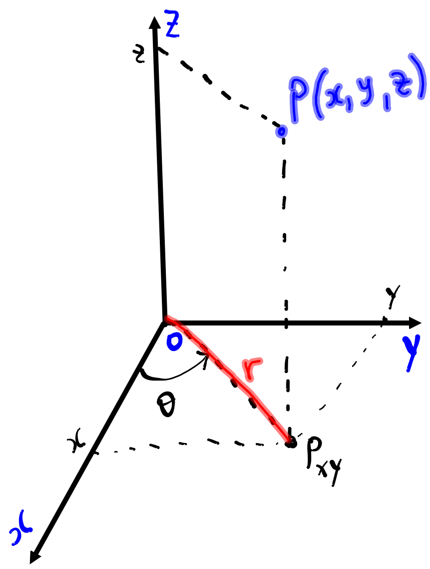
13.9-13.10: Part I

Triple integrals in cylindrical coordinates

• **Cylindrical coordinates:**



$P(x, y, z) \in \mathbb{R}^3$

In the cylindrical coordinates P is represented by the ordered triple (r, θ, z), where r, θ are the polar coordinates of P_{xy} and z is the directed distance from the xy-plane to P:

$$\underbrace{x = r\cos\theta \qquad y = r\sin\theta}_{\text{polar coordinates}} \qquad z = z$$

where

$$r^2 = x^2+y^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z.$$

attention
what quadrant

REMARK 1. The cylindrical coordinates

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \\ r &\geq 0, \quad 0 \leq \theta \leq 2\pi \end{aligned}}$$

are useful in problems that involve *symmetry about the z-axis.*

EXAMPLE 2. *Find an equation in cylindrical coordinates for the cone*

$$z = \sqrt{x^2 + y^2}$$

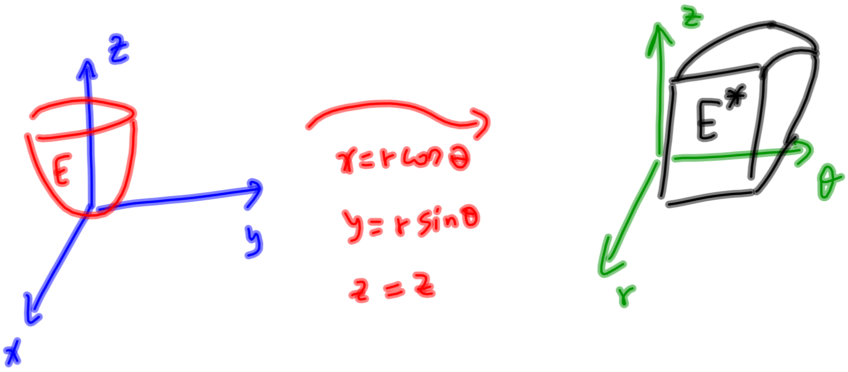
$$z = \sqrt{r^2} \Rightarrow \boxed{z = r}$$

THEOREM 3. *Let $f(x, y, z)$ be a continuous function over a solid $E \subset \mathbb{R}^3$. Let E^* be its image in cylindrical coordinates. Then*

$$\iiint_E f(x, y, z)\, \mathrm{d}V = \iiint_{E^*} f(r\cos\theta, r\sin\theta, z)\, \mathrm{d}V^*,$$

where

$$\mathrm{d}V^* = r\, \mathrm{d}r\, \mathrm{d}z\, \mathrm{d}\theta.$$

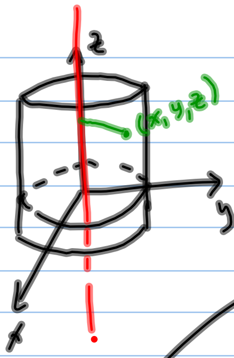


EXAMPLE 4. *The density at any point of the solid E,*

$$\boxed{E = \{(x, y, z) : x^2 + y^2 \le 9, -1 \le z \le 4\}}$$

equals to its distance from the axis of E. Find the mass of E.

z- axis



$$\rho(x,y,z) = \sqrt{x^2+y^2}$$

$$m = \iiint_E \rho \, dV = \iiint_E \sqrt{x^2+y^2} \, dV$$

cylindrical coordinates

$$E^* = \{(r,\theta,z) : r^2 \le 9, \ -1 \le z \le 4\}$$

$$E^* = \{(r,\theta,z) : 0 \le r \le 3, \ -1 \le z \le 4, \ 0 \le \theta \le 2\pi\}$$

$$m = \iiint_{E^*} r \, dV^* = \int_0^{2\pi}\int_0^3\int_{-1}^4 r \, r \, dz \, dr \, d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^3 r^2 \, dr\right) \cdot \left(\int_{-1}^4 dz\right)$$

$$= 2\pi \left.\frac{r^3}{3}\right|_0^3 \cdot 5 = 10\pi \cdot \frac{3^3}{3} = \boxed{90\pi}$$

REMARK 5. If E is a solid region of type I, i.e.

$$E = \{(x, y, z) | (x, y) \in D, \phi_1(x, y) \leq z \leq \phi_2(x, y)\},$$

where D is the projection of E onto the xy-plane then, as we know,

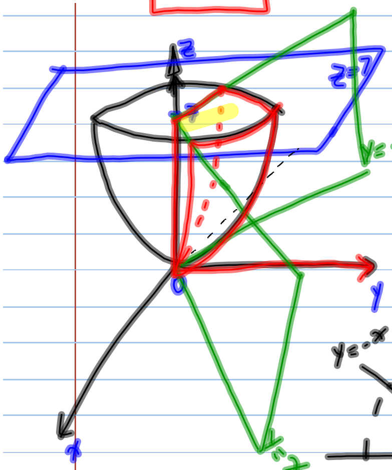
$$\iiint_E f(x, y, z)\,\mathrm{d}V = \iint_D \left[\int_{\phi_1(x,y)}^{\phi_2(x,y)} f(x, y, z)\,\mathrm{d}z\right]\mathrm{d}A.$$

Passing to cylindrical coordinates here we actually have to replace D by its image D^* in polar coordinates and $\mathrm{d}z\,\mathrm{d}A$ by $r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$.

EXAMPLE 6. *Find the volume of the solid E bounded by the surfaces*

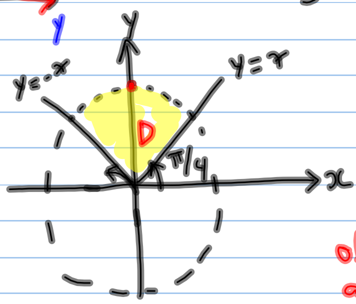
$$\underbrace{y = x, \quad y = -x,}_{\text{planes}} \quad \underbrace{x^2 + y^2 = 5z,}_{\text{paraboloid}} \quad \underbrace{z = 7}_{\text{plane}}$$

so that $\boxed{y \geq 0}$.



$$V = \iiint_E dV$$

$$E = \left\{ \frac{x^2+y^2}{5} \leq z \leq 7, \quad (x,y) \in D \right\}$$



To find radius of the circle which is boundary of D we find the line of intersection of paraboloid and plane $z=7$

$$\begin{cases} x^2+y^2 = 5z \\ z = 7 \end{cases} \Rightarrow x^2+y^2 = 35$$

radius $= \sqrt{35}$

Way I

$$V = \iiint_E dV = \iint_D \left(\int_{\frac{x^2+y^2}{5}}^{7} dz \right) dA = \iint_D \left(7 - \frac{x^2+y^2}{5}\right) dA \rightarrow \text{use polar coordinates}$$

Way II Use directly cylindrical coordinates:

$$E^* = \left\{ (r,\theta,z) : \frac{r^2}{5} \leq z \leq 7, \ 0 \leq r \leq \sqrt{35}, \ \frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4} \right\}$$

$$V = \iiint_E dV = \iiint_{E^*} dV^* = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_0^{\sqrt{35}} \int_{\frac{r^2}{5}}^{7} r \, dz \, dr \, d\theta = \left(\int_{\frac{\pi}{4}}^{3\pi/4} d\theta \right) \cdot \int_0^{\sqrt{35}} r \cdot z \Big|_{z=\frac{r^2}{5}}^{7} dr$$

$$= \left(\frac{3\pi}{4} - \frac{\pi}{4} \right) \int_0^{\sqrt{35}} \left(7r - r\frac{r^2}{5} \right) dr = \frac{\pi}{2} \int_0^{\sqrt{35}} \left(7r - \frac{r^3}{5} \right) dr$$

$$= \frac{\pi}{2} \left(\frac{7r^2}{2} \Big|_0^{\sqrt{35}} - \frac{r^4}{20} \Big|_0^{\sqrt{35}} \right) = \ldots = \frac{245\pi}{8}$$