14.6: Parametric surfaces and their areas

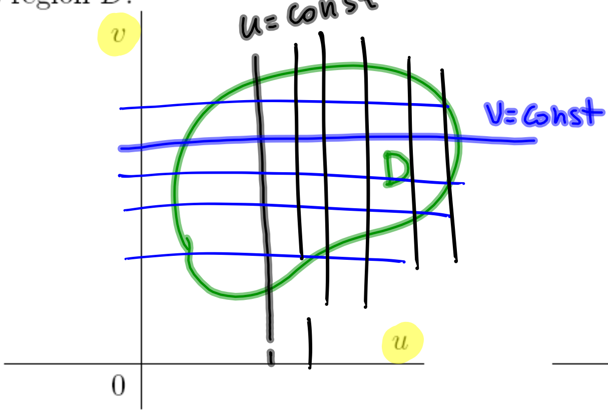
Consider a continuous vector valued function of two variables

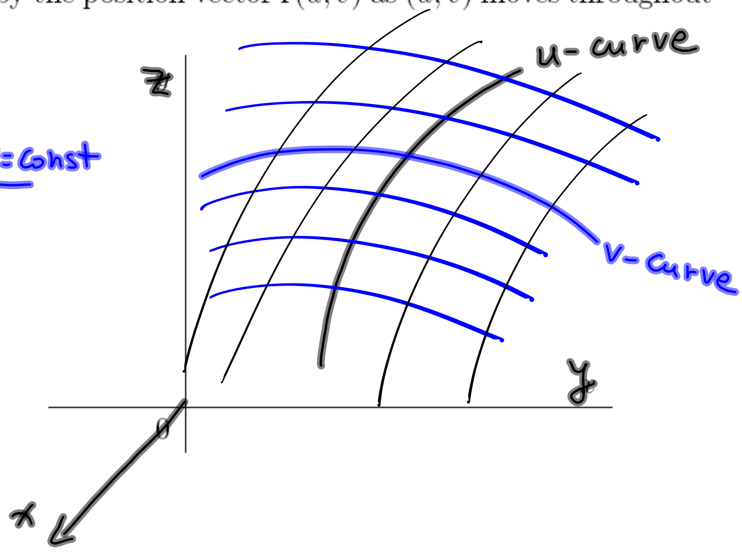
$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}, \quad (u,v) \in D.$$

Parametric surface:

$$S: \; x = x(u,v), \quad y = y(u,v), \quad z = z(u,v), \quad (u,v) \in D.$$

In other words, the surface S is traces out by the position vector $\mathbf{r}(u,v)$ as (u,v) moves throughout the region D.





Find graph

EXAMPLE 1. *Determine the surface given by the parametric representation*

$$\mathrm{r}(u,v) = \langle u, u\cos v, u\sin v\rangle\,, \quad 1 \le u \le 5, \quad 0 \le v \le 2\pi$$

$x = u$
$y = u\cos v$
$z = u\sin v$

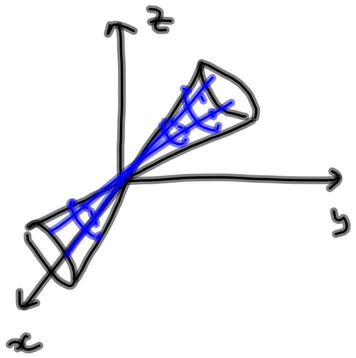
Eliminate v:

$y^2 + z^2 = u^2(\underbrace{\cos^2 v + \sin^2 v}_{1}) = u^2$

Eliminate u

x^2

$x^2 = y^2 + z^2$
cone



EXAMPLE 2. *Give parametric or vector representations for each of the following surfaces (indicate the domain of the parameters):*

(a) *cylinder:* $x^2 + y^2 = 9, \quad 1 \leq z \leq 5.$

Use cylindrical coordinates:

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}\right\} \quad \begin{aligned} x^2+y^2 &= 9 \\ r^2 &= 9 \Rightarrow r = 3 \end{aligned} \Rightarrow \boxed{\begin{aligned} x &= 3\cos\theta \\ y &= 3\sin\theta \\ z &= z \end{aligned}}$$

$$\vec{r}(\theta, z) = \langle 3\cos\theta, 3\sin\theta, z \rangle$$

$$D = \{ (\theta, z) : 1 \leq z \leq 5, \ 0 \leq \theta \leq 2\pi \}$$

$\Rightarrow x^2+y^2+z^2 = 100, \ z \geq 0$

(b) *upper half-sphere:* $z = \sqrt{100 - x^2 - y^2}.$

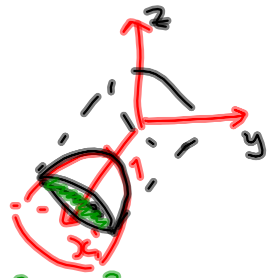
1. $\vec{r}(x, y) = \langle x, y, \sqrt{100 - x^2 - y^2} \rangle, \quad D = \{ x^2 + y^2 \leq 100 \}$

2. Use spherical coordinates $\Rightarrow \rho = 10$

$$x = 10 \sin\varphi \cos\theta$$
$$y = 10 \sin\varphi \sin\theta$$
$$z = 10 \cos\varphi$$

$$D = \{ (\theta, \varphi) : 0 \leq \theta \leq 2\pi, \ 0 \leq \varphi \leq \tfrac{\pi}{2} \}$$

(c) *elliptic paraboloid:* $x = 3y^2 + z^2 + 1$, *where* $1 \le x \le 2$.

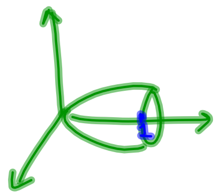


$\vec{r}(y,z) = \langle 3y^2 + z^2 + 1,\ y,\ z \rangle$

To find D we need projection onto the yz-plane: $3y^2 + z^2 + 1 = 2 \Rightarrow 3y^2 + z^2 = 1$

$D = \{ 3y^2 + z^2 \le 1 \}$

(d) *elliptic paraboloid* $y = \overbrace{x^2 + 4z^2}^{=r^2}$, *where* $0 \le y \le 1$



$$\boxed{\begin{aligned} x &= x \\ y &= x^2 + 4z^2 \\ z &= z \end{aligned}}$$

$\{ x^2 + 4z^2 \le 1 \} = D$

2nd solution

use polar coordinates

$$\left.\begin{aligned} x &= r\cos\theta \\ 2z &= r\sin\theta \end{aligned}\right\} \quad y = x^2 + 4z^2 = x^2 + (2z)^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2$$

$$\boxed{x = r\cos\theta, \quad y = r^2, \quad z = \frac{r}{2}\sin\theta}$$

where $D = \{ (r,\theta) : 0 \le r \le 1,\ 0 \le \theta \le 2\pi$

CONCLUSION: To parametrize surface we may use polar, cylindrical or spherical coordinates, or

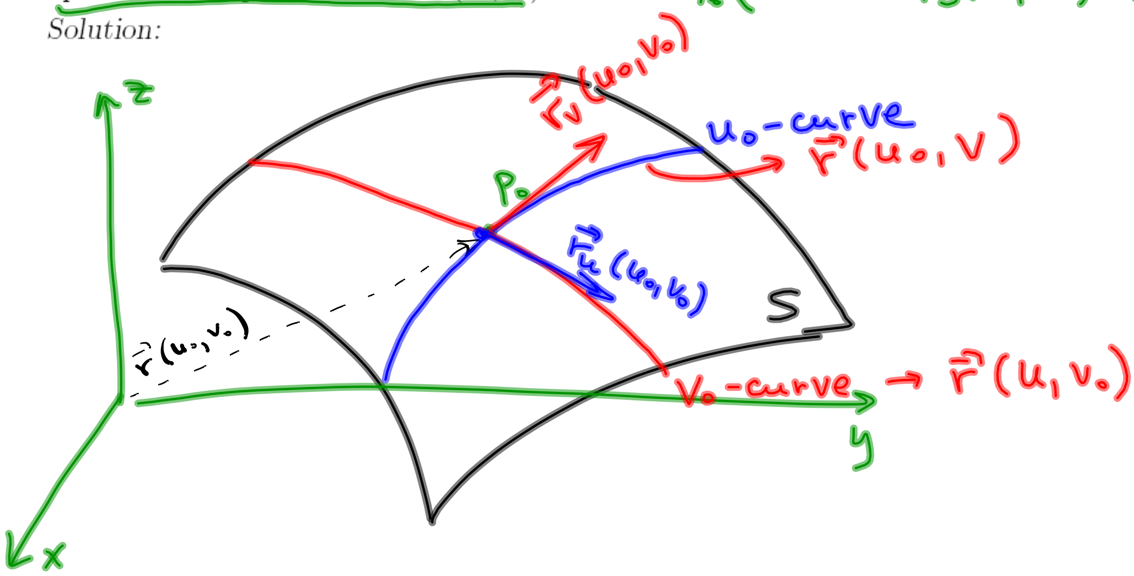
- $z = f(x,y) \longrightarrow \mathbf{r}(x,y) = x\mathbf{i} + y\mathbf{j} + f(x,y)\mathbf{k}$
- $y = f(x,z) \longrightarrow \mathbf{r}(x,z) = x\mathbf{i} + f(x,z)\mathbf{j} + z\mathbf{k}$
- $x = f(y,z) \longrightarrow \mathbf{r}(y,z) = f(y,z)\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

when a surface is the graph of

- **Tangent planes:**

PROBLEM: Find the tangent plane to a parametric surface S given by a vector function $\mathbf{r}(u, v)$ *at a point P_0 with position vector* $\mathbf{r}(u_0, v_0)$. $\Rightarrow$ $P_0\left(x(u_0,v_0), y(u_0,v_0), z(u_0,v_0)\right)$

Solution:



The **normal vector**

$$\mathbf{N} = \mathbf{N}(u, v) = \vec{r}_u \times \vec{r}_v$$

If the normal vector is not $\mathbf{0}$ then the surface S is called **smooth** (it has no "corner").

Special Case: a surface S given ~~by a~~ as a *graph* $z = f(x, y)$. Then one can choose the following parametrization of S:

$$\mathrm{r}(x, y) = \langle x, y, f(x,y) \rangle$$

and the then the normal vector is

$$\mathrm{N} = \vec{N}(x,y) = \vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix}$$

$$= \hat{i}\,(0 - f_x) - \hat{j}\,(f_y - 0) + \hat{k}\cdot(1-0)$$

$$= \langle -f_x, -f_y, 1 \rangle$$

EXAMPLE 3. *Find the tangent plane to the surface with parametric equations* $x = uv + 1, y = ue^v, z = ve^u$ *at the point* $(1, 0, 0) = (x_0, y_0, z_0)$

$x = uv + 1 = 1$

$y = ue^v = 0 \Rightarrow u = 0$

$z = ve^u = 0$

$v \cdot e^0 = 0 \Rightarrow v = 0$

$0 \cdot 0 + 1 = 1$ (True)

$(u_0, v_0) = (0, 0)$

Tangent plane $A(x-1) + B(y-0) + C(z-0) = 0$

where $\langle A, B, C \rangle = \vec{N}(u_0, v_0) = \vec{N}(0,0) = \vec{r}_u(0,0) \times \vec{r}_v(0,0)$

$\vec{r}(u,v) = \langle uv+1, ue^v, ve^u \rangle$

$\vec{r}_u = \langle v, e^v, ve^u \rangle \Rightarrow \vec{r}_u(0,0) = \langle 0, 1, 0 \rangle$

$\vec{r}_v = \langle u, ue^v, e^u \rangle \Rightarrow \vec{r}_v(0,0) = \langle 0, 0, 1 \rangle$

$$\vec{N}(0,0) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle \underset{A}{1}, \underset{B}{0}, \underset{C}{0} \rangle$$

$$\boxed{x = 1}$$

• **Surface Area:**

Consider a smooth surface S given by

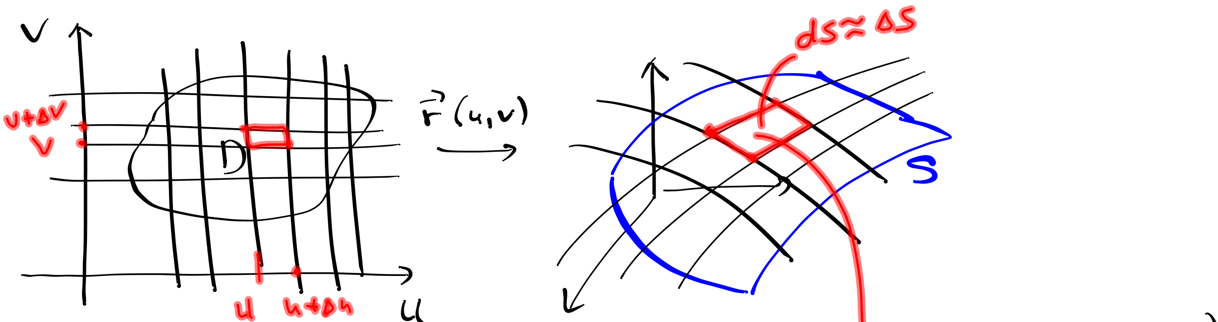
$$\mathbf{r}(u,v)=x(u,v)\mathbf{i}+y(u,v)\mathbf{j}+z(u,v)\mathbf{k},\quad (u,v)\in D,$$

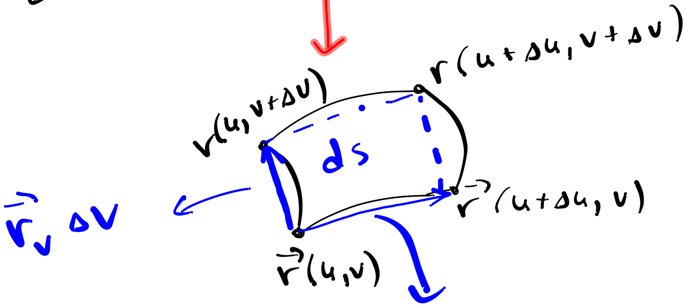
then

$$\mathrm{d}S = |N(u,v)|\mathrm{d}u\mathrm{d}v = |\vec{r}_u \times \vec{r}_v|dudv$$

and the surface area

$$A(S)=\underbrace{\iint_S \mathrm{d}S}_{\text{surface integral}} = \overbrace{\iint_D |\mathbf{r}_u\times\mathbf{r}_v|\,\mathrm{d}A}^{\text{double integral}}.$$





$$\begin{aligned} ds &= |(\vec{r}_u \Delta u)\times(\vec{r}_v \Delta v)| \\ &= |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v \\ &= |\vec{r}_u \times \vec{r}_v| dudv = |\vec{N}(u,v)|dudv \end{aligned}$$

$$\vec{r}(u+\Delta u, v)-\vec{r}(u,v) \approx \vec{r}_u \Delta u$$

normal $\vec{N} = \langle f_x, f_y, -1\rangle$

REMARK 4. *Special Case:* a surface S given by a *graph* $z=f(x,y)$ we have

$$\mathbf{r}(x,y)=x\mathbf{i}+y\mathbf{j}+f(x,y)\mathbf{k}$$

and

$$\mathrm{d}S = |\mathbf{N}(x,y)|\,\mathrm{d}A = \sqrt{(f_x)^2+(f_y)^2+1}\;dxdy$$

EXAMPLE 5. *Find the surface area of the surface*

$$S: \quad x = uv, \quad y = u + v, \quad z = u - v, \quad \left\{u^2 + v^2 \leq 1.\right\} = D$$

$$A(S) = \iint_S dS = \iint_D |\vec{r}_u \times \vec{r}_v| \, du\,dv$$

$$\vec{r}(u,v) = \langle uv, u+v, u-v \rangle$$

$$\vec{r}_u = \langle v, 1, 1 \rangle \Rightarrow \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v & 1 & 1 \\ u & 1 & -1 \end{vmatrix} = -2\hat{i} + (v+u)\hat{j} + (v-u)\hat{k}$$

$$\vec{r}_v = \langle u, 1, -1 \rangle$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{(-2)^2 + (v+u)^2 + (v-u)^2}$$

$$= \sqrt{4 + v^2 + u^2 + 2\cancel{uv} + v^2 + u^2 - 2\cancel{uv}} = \sqrt{4 + 2(u^2+v^2)}$$

$$A(S) = \iint_{D=\{u^2+v^2 \leq 1\}} \sqrt{4 + 2(u^2+v^2)} \, \underbrace{du\,dv} \Rightarrow \text{use polar coord.}$$

$u = r\cos\theta$

$v = r\sin\theta$

$D = \{(r,\theta) : 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$

$$A(S) = \int_0^{2\pi}\int_0^1 \sqrt{4+2r^2}\, r\,dr\,d\theta = 2\pi \int_0^1 r\sqrt{4+2r^2}\,dr$$

u-subst. $\quad$ $\parallel$ $\quad$ . . . .

EXAMPLE 6. *Find the surface area of the part paraboloid $z = x^2 + y^2$ between two planes: $z = 0$ and $z = 4$.*

$\underbrace{x = x, \quad y = y,}_{\text{parameters}} \quad \underbrace{z = x^2 + y^2}$

Find parameter

$0 \le z \le 4$

$D = \{ 0 \le x^2 + y^2 \le 4 \}$

$\vec{n}(x,y) = \langle z_x, z_y, -1 \rangle = \langle 2x, 2y, -1 \rangle$

$|\vec{n}(x,y)| = \sqrt{4(x^2+y^2)+1}$

$dS = |\vec{n}|\, dA = \sqrt{4(x^2+y^2)+1}\, dA$

$$SA = \iint_S dS = \iint_{D=\{x^2+y^2 \le 4\}} \sqrt{4(x^2+y^2)+1}\, dA$$

use polar coordinates

$$= \int_0^{2\pi} \left(\int_0^2 \sqrt{4r^2+1}\; r\, dr \right) d\theta$$

$$= 2\pi \int_0^2 r\sqrt{4r^2+1}\, dr$$

u-sub

$u = 4r^2 + 1$

$du = 8r\, dr$

$1 \le u \le 17$

$$= \frac{\pi}{4} \int_1^{17} \sqrt{u}\, du = \frac{\pi}{4} \cdot \frac{2}{3} u\sqrt{u} \Big|_1^{17}$$

$$= \frac{\pi}{6} \left(17\sqrt{17} - 1 \right)$$