Section 1.3: Vector functions

Parametric equations:

$$x = x(t), \quad y = y(t) \quad , \qquad a \leq t \leq b$$

where the variable t is called a **parameter.** Each value of the parameter t defines a point that we can plot. As t varies over its domain we get a collection of points $(x, y) = (x(t), y(t))$ on the plane which is called the **parametric curve**.

Each parametric curve can be represented as the **vector function**:

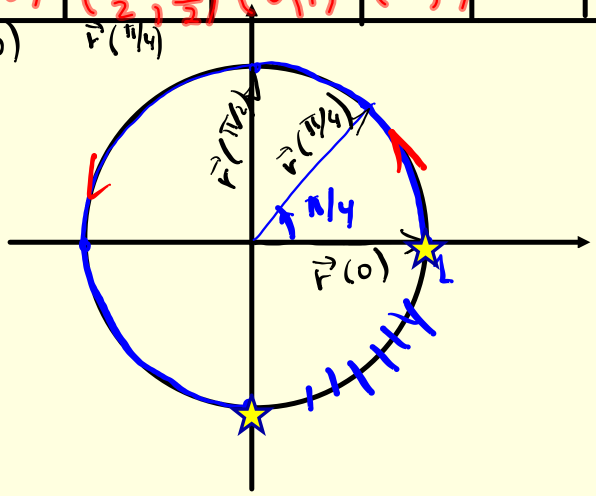
$$\overrightarrow{r(t)} = \langle x(t), y(t) \rangle \, .$$

Note that Parametric curves have a **direction of motion** given by increasing of parameter t. So, when sketching parametric curves we also include arrows that show the direction of motion.

$\vec{r}(t) = \langle \cos t, \sin t \rangle$

EXAMPLE 1. *Examine the parametric curve* $x = \cos t, \quad y = \sin t, \quad \underbrace{0 \le t \le 3\pi/2}_{\text{parameter's domain}}$.

t	0	$\pi/4$	$\pi/2$	π	...	$3\pi/2$
(x,y)	$(1,0)$	$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$	$(0,1)$	$(-1,0)$	...	$(0,-1)$
	$\vec{r}(0)$	$\vec{r}(\pi/4)$				



EXAMPLE 2. *Given* $\mathbf{r}(t) = \langle t+1, t^2 \rangle$. (or $x = t+1$, $y = t^2$)

(a) *Does the point* $(4, 3)$ *belong to the graph of* $\mathbf{r}(t)$*?*

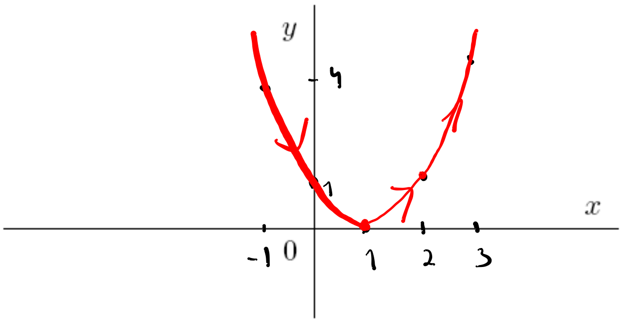
Find t such that $\vec{r}(t) = \langle 4, 3 \rangle \Rightarrow \langle t+1, t^2 \rangle = \langle 4, 3 \rangle \Rightarrow$

$$\left.\begin{array}{l} t+1 = 4 \Rightarrow t = 3 \\ t^2 = 3 \Rightarrow t = \pm\sqrt{3} \end{array}\right\} \text{IMPOSSIBLE}$$

NO

(b) *Sketch the graph of* $\mathbf{r}(t)$.

t	$\mathbf{r}(t)$
-2	$\langle -1, 4 \rangle$
-1	$\langle 0, 1 \rangle$
0	$\langle 1, 0 \rangle$
1	$\langle 2, 1 \rangle$
2	$\langle 3, 4 \rangle$



(c) *Find the Cartesian equation of* $\mathbf{r}(t)$ *eliminating the parameter.*

$$\begin{cases} x = t+1 \\ y = t^2 \end{cases}$$ Eliminate parameter t

$t = x - 1$

$y = (x-1)^2$ parabola

EXAMPLE 3. *Find the Cartesian equation for* $\mathbf{r}(t) = \cos t\,\mathbf{i} + \cos(2t)\,\mathbf{j}$

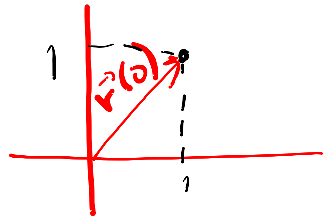
Elimination of parameter

$$\begin{cases} x = \cos t \\ y = \cos 2t = 2\cos^2 t - 1 \end{cases} \Rightarrow \underbrace{\boxed{y = 2x^2 - 1}}_{\text{parabola}}$$

EXAMPLE 4. *An object is moving in the xy-plane and its position after t seconds is given by* $\mathbf{r}(t) = \langle 1+t^2, 1+3t\rangle$.

(a) *Find the position of the object at time $t = 0$.*

$\vec{r}(0) = \langle 1, 1\rangle$



(b) *At what time does the object reach the point* $(10, 10)$.

Find t s.t. $\vec{r}(t) = \langle 10, 10\rangle$

$\langle 1+t^2, 1+3t\rangle = \langle 10, 10\rangle$

$\begin{cases} 1+t^2 = 10 \Rightarrow t^2 = 9 \Rightarrow t = \pm 3 \\ 1+3t = 10 \Rightarrow t = 3 \end{cases} \Rightarrow \boxed{t = 3}$

(c) *Does the object pass through the point* $(20, 20)$*?*

$1+t^2 = 20 \rightarrow \begin{cases} t = \pm\sqrt{19} \\ t = \frac{19}{3} \end{cases}$ impossible

$1+3t = 20$

NO

(d) *Find an equation in x and y whose graph is the path of the object.*

$\vec{r}(t) = \langle 1+t^2, 1+3t\rangle$

Eliminate parameter

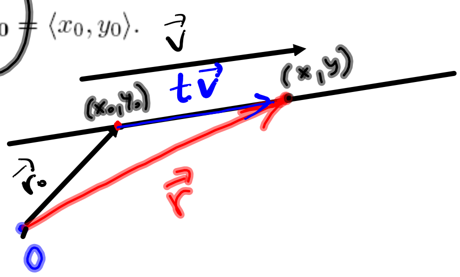
$\begin{cases} x = 1+t^2 \\ y = 1+3t \Rightarrow t = \frac{y-1}{3} \end{cases}$

$$\boxed{x = 1 + \left(\frac{y-1}{3}\right)^2}$$

A **Vector equation of the line** passing through the point (x_0, y_0) and parallel to the vector $\mathbf{v} = \langle a, b\rangle$ is given by

$$\mathbf{r} = \mathbf{r_0} + t\mathbf{v},$$

where $\mathbf{r_0} = \langle x_0, y_0\rangle$.



$\vec{r} = \langle x, y\rangle$

$\vec{r} = \vec{r}_0 + t\vec{v}$

The vector equation of the line can be rewritten in parametric form. Namely, we have

$$\begin{aligned} \vec{r}(t) = \langle x(t), y(t)\rangle &= \mathbf{r} = \mathbf{r_0} + t\mathbf{v} = \\ &= \langle x_0, y_0\rangle + t\,\langle a, b\rangle = \langle x_0, y_0\rangle + \langle ta, tb\rangle = \\ &= \langle x_0 + ta, y_0 + tb\rangle\,. \end{aligned}$$

This immediately yields that the **parametric equations of the line** passing through the point (x_0, y_0) and parallel to the vector $\mathbf{v} = \langle a, b\rangle$ are

$$\boxed{x(t) = x_0 + at, \quad y(t) = y_0 + bt.}$$

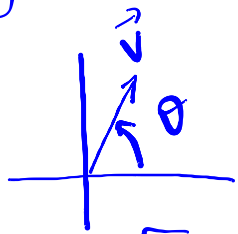
EXAMPLE 5. *Find parametric equations of the line*

(a) *passing through the point* $(1, 0)$ *and parallel to the vector* $\mathbf{i} - 4\mathbf{j}$;

$(x_0, y_0) = (1, 0)$ $\qquad \vec{v} = \langle 1, -4 \rangle$

$$\boxed{x = 1 + t, \quad y = -4t}$$

(b) *passing through the point* $(-4, 5)$ *with slope* $\sqrt{3}$;



$(x_0, y_0) = (-4, 5)$

$\vec{v}$ - ? $\qquad \vec{v} = \langle \cos\theta, \sin\theta \rangle =$

$= \langle \cos\frac{\pi}{3}, \sin\frac{\pi}{3} \rangle = \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$

$\tan\theta = \sqrt{3}$

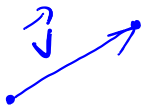
$\theta = \frac{\pi}{3}$

$$\boxed{\begin{aligned} x &= -4 + \tfrac{1}{2}t \\ y &= 5 + \tfrac{\sqrt{3}}{2}t \end{aligned}}$$

(c) *passing through the points* $(7, 2)$ *and* $(3, 2)$.

$(x_0, y_0) = (7, 2)$

$\vec{v} = \langle 7, 2 \rangle - \langle 3, 2 \rangle = \langle 4, 0 \rangle$



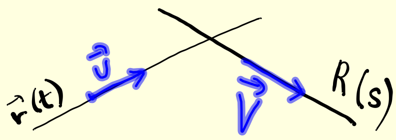
$$\boxed{\begin{aligned} x &= 7 + 4t \\ y &= 2 \end{aligned}}$$

the vectors $\vec{v}$ and $\vec{V}$

EXAMPLE 6. *Determine whether the lines* $\mathbf{r}(t) = \langle 1+t, 1-3t\rangle$, $\mathbf{R}(s) = \langle 1+3s, 12+s\rangle$ *are parallel, orthogonal or neither. If they are not parallel, find the intersection point.*

$\vec{r}(t) = \langle 1+t, 1-3t\rangle$ $\qquad$ $R(s) = \langle 1+3s, 12+s\rangle$

$\vec{v} = \langle 1, -3\rangle$ $\qquad$ $\vec{V} = \langle 3, 1\rangle$



$\vec{v} \neq c\vec{V}$ or $\frac{1}{3} \neq \frac{-3}{1}$ $\Rightarrow$ $\vec{v} \nparallel \vec{V} \Rightarrow$ the lines are not parallel

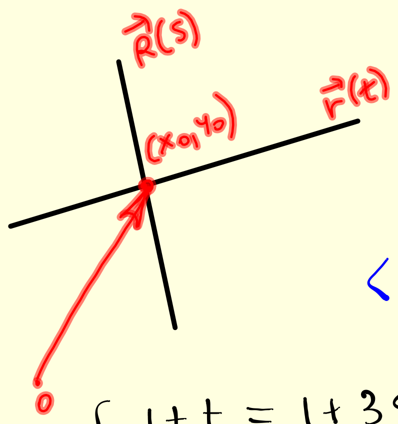
Note that $\vec{v} = \vec{V}^{\perp}$ $\longrightarrow$ $\vec{v} \perp \vec{V} \Rightarrow$ the lines are orthogonal

or $\vec{v} \cdot \vec{V} = \langle 1, -3\rangle \cdot \langle 3, 1\rangle = 3 - 3 = 0$

Find intersection point

There are values of s and t such that

$$\underline{\vec{r}(t) = \vec{R}(s)} = \boxed{\langle x_0, y_0\rangle}$$



$$\langle 1+t, 1-3t\rangle = \langle 1+3s, 12+s\rangle$$

$$\begin{cases} 1+t = 1+3s \Rightarrow t = 3s \\ 1-3t = 12+s \Rightarrow s+3t = -11 \end{cases}$$

$s + 3 \cdot 3s = -11$

$10s = -11$

$s = -1.1$

$t = 3s = -3.3$

$\vec{r}(-3.3) = R(-1.1) = \langle 1 - 3.3, 1 + 3 \cdot 3.3\rangle =$

$= \langle -2.3, 10.9\rangle$

$\boxed{(-2.3, 10.9)}$