3.4 Derivatives of trigonometric functions

It is important to remember that everything for six trigonometric functions ($\sin x, \cos x, \tan x, \cot x, \csc x, \sec x$) will be done in radians.

EXAMPLE 1. *Compute:*

(a) $\lim_{x\to 0} \sin x = \sin 0 = 0$ (b) $\lim_{x\to 0} \cos x = \cos 0 = 1$

THEOREM 2.

$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1,} \qquad \boxed{\lim_{x\to 0} \frac{\cos x - 1}{x} = 0.}$$

Proof

$$\lim_{x\to 0} \frac{\cos x - 1}{x} = \lim_{x\to 0} \frac{\cos x - 1}{x} \cdot \frac{\cos x + 1}{\cos x + 1} = \lim_{x\to 0} \frac{\cos^2 x - 1}{x(\cos x + 1)}$$

$$= \lim_{x\to 0} \frac{-(1-\cos^2 x)}{x(\cos x + 1)} = -\lim_{x\to 0} \frac{\sin^2 x}{x^2} \cdot \frac{x}{\cos x + 1}$$

$$= -\underbrace{\lim_{x\to 0} \left(\frac{\sin x}{x}\right)^2}_{1^2} \cdot \underbrace{\lim_{x\to 0} \frac{x}{\cos x + 1}}_{\frac{0}{1+1}} = 1 \cdot 0 = 0$$

EXAMPLE 3. *Find these limits:*

(a) $\lim_{x\to 0} \frac{\sin(5x)}{x}$ $= \lim_{x\to 0} \frac{\sin(5x)}{5x}\cdot 5 =$ $y=5x \to 0$, $x\to 0$

$= \lim_{y\to 0}\left(5\,\frac{\sin y}{y}\right) = 5 \lim_{y\to 0} \frac{\sin y}{y} = 5\cdot 1 = \boxed{5}$

2nd way $y = 5x \to 0$, $x\to 0$, $x = \frac{y}{5}$ $\lim_{x\to 0} \frac{\sin(5x)}{x} = \lim_{y\to 0} \frac{\sin y}{y/5} = 5 \lim_{y\to 0} \frac{\sin y}{y}$

$= 5\cdot 1 = \boxed{5}$

(b) $\lim_{x\to 0} \frac{\sin(9x)}{\sin(7x)}$ $= \lim_{x\to 0} \underbrace{\frac{\sin(9x)}{9x}}_{1} \cdot \underbrace{\frac{7x}{\sin(7x)}}_{1} \cdot \underbrace{\frac{9x}{7x}}_{\frac{9}{7}} = \boxed{\frac{9}{7}}$

(c) $\lim_{x\to 0} \frac{x}{\sin(4x)}$ $= \lim_{y\to 0} \frac{y/4}{\sin y} = \frac{1}{4} \lim_{y\to 0} \frac{y}{\sin y} = \boxed{\frac{1}{4}}$

$y = 4x \to 0$, $x\to 0$

$x = \frac{y}{4}$

$\lim_{x\to 0} \frac{\sin(ax)}{x} = a$, $\lim_{x\to 0} \frac{x}{\sin(ax)} = \frac{1}{a}$, $\lim_{x\to 0} \frac{\sin(ax)}{\sin(bx)} = \frac{a}{b}$

Note $\lim_{x\to 0} \frac{x}{\sin x} = 1$ Remind $\lim_{x\to 0} \frac{\sin x}{x} = 1$

(d) $\lim_{x\to 0} \frac{1}{x^2 \cot^2(3x)} = \lim_{x\to 0} \frac{1}{x^2 \frac{\cos^2(3x)}{\sin^2(3x)}} = \lim_{x\to 0} \underbrace{\left(\frac{\sin(3x)}{x}\right)^2}_{3^2} \cdot \underbrace{\frac{1}{\cos^2 3x}}_{1} = 9$

(e) $\lim_{x\to 0} \frac{\cos x - 1}{\sin x} = \lim_{x\to 0} \underbrace{\frac{\cos x - 1}{x}}_{\downarrow \atop 0} \cdot \underbrace{\frac{x}{\sin x}}_{\downarrow \atop 1} = 0 \cdot 1 = 0$

EXAMPLE 4. *Find the following derivatives:*

(a) $\dfrac{\mathrm{d}}{\mathrm{d}x}\sin x = \lim\limits_{h\to 0} \dfrac{\sin(x+h) - \sin x}{h} = \lim\limits_{h\to 0} \dfrac{\sin x \cos h + \sin h \cos x - \sin x}{h}$

$$= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \sin h \cos x}{h}$$

$$= \sin x \underbrace{\lim_{h\to 0} \frac{\cos h - 1}{h}}_{\to 0} + \cos x \underbrace{\lim_{h\to 0} \frac{\sin h}{h}}_{1} = \cos x$$

Remark Similarly one can get $(\cos x)' = -\sin x$.

(b) $\dfrac{\mathrm{d}}{\mathrm{d}x}\tan x = \dfrac{d}{dx}\left(\dfrac{\sin x}{\cos x}\right) = \dfrac{f'g - fg'}{g^2}$

$$= \frac{(\sin x)' \cos x - \sin x (\cos x)'}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

Derivatives of Trig Functions (memorize these!)

$\dfrac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x$	$\dfrac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x$	$\dfrac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x$
$\dfrac{\mathrm{d}}{\mathrm{d}x}\csc x = -\csc x \cot x$	$\dfrac{\mathrm{d}}{\mathrm{d}x}\sec x = \sec x \tan x$	$\dfrac{\mathrm{d}}{\mathrm{d}x}\cot x = -\csc^2 x$

EXAMPLE 5. *Find the derivative of these functions.*

(a) $y = \cot x + 5\sec x + x\sqrt{x}$ $= x^{3/2}$

$y' = -\csc^2 x + 5 \sec x \tan x + \frac{3}{2}\sqrt{x}$

(b) $f'(x) = \left(\frac{\cos x}{1+\sin x}\right)' = \frac{(\cos x)'(1+\sin x) - \cos x\,(1+\sin x)'}{(1+\sin x)^2}$

$= \frac{-\sin x\,(1+\sin x) - \cos x \cos x}{(1+\sin x)^2}$

$= \frac{-\sin x - (\sin^2 x + \cos^2 x)}{(1+\sin x)^2} = \frac{-\sin x - 1}{(1+\sin x)^2}$

$= -\frac{(\sin x + 1)}{(1+\sin x)^2} = -\frac{1}{1+\sin x}$

EXAMPLE 6. *Find the equation of the tangent line to the graph of function* $\underbrace{y = x^2 \sin x}$ *at* $x = \frac{\pi}{4}$.

$$y - y\left(\frac{\pi}{4}\right) = y'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right)$$

$$y\left(\frac{\pi}{4}\right) = \left(\frac{\pi}{4}\right)^2 \cdot \sin\frac{\pi}{4} = \frac{\pi^2}{16} \cdot \frac{\sqrt{2}}{2} = \frac{\pi^2\sqrt{2}}{32}$$

$$y'(x) = (x^2 \sin x)' = 2x \sin x + x^2 \cos x$$

$$y'\left(\frac{\pi}{4}\right) = \cancel{2} \cdot \frac{\pi}{4}\, \frac{\sqrt{2}}{\cancel{2}} + \frac{\pi^2}{16} \cdot \frac{\sqrt{2}}{2} = \frac{\pi}{4}\sqrt{2}\left(1 + \frac{\pi}{8}\right)$$ slope

$$y - \frac{\pi^2\sqrt{2}}{32} = \frac{\pi\sqrt{2}}{4}\left(1 + \frac{\pi}{8}\right)\left(x - \frac{\pi}{4}\right)$$

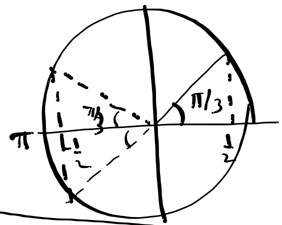
Ex. 7 Find where the tangent line to the graph of $y = x + 2\sin x$ is horizontal.

$\Downarrow$

Find x such that $y'(x) = 0$

$$y'(x) = 1 + 2\cos x = 0$$

$$\cos x = -\frac{1}{2}$$



$$\boxed{x = \pi \pm \frac{\pi}{3} + 2\pi k, \text{ where } k \text{ is integer}}$$