\3.6: **Implicit differentiation**

EXAMPLE 1. *Find y' if the $y = y(x)$ satisfies the equation $xy = 5$ for all values of x in its domain and evaluate $y'(5)$.*

`Solution 1` (by explicit differentiation):

$$xy=5 \Rightarrow y = \frac{5}{x} \Rightarrow y' = -\frac{5}{x^2}$$

$$y'(5) = -\frac{5}{5^2} = -\frac{1}{5}$$

`Solution 2` (by implicit differentiation):

$$x\,y(x) = 5$$

$$\frac{d}{dx}(x\,y(x)) = \frac{d}{dx}(5)$$

$$x'y + xy' = 0$$

$$y + xy' = 0$$

$$xy' = -y \Rightarrow y' = -\frac{y}{x}$$

If $x=5$ $\Downarrow$ $y=1$

$$y'(5) = -\frac{1}{5}$$

EXAMPLE 2. (a) *If* $x^2 + y^2 = 16$ *find* $\frac{dy}{dx}$.

$y = y(x)$

$$\frac{d}{dx}\left[x^2 + [y(x)]^2\right] = \frac{d}{dx}(16)$$

$$2x + 2y(x)\frac{dy}{dx} = 0$$

$$2y(x)\frac{dy}{dx} = -2x \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

(b) *Find the equation of the tangent line to* $x^2 + y^2 = 16$ *at the point* $(2, 2\sqrt{3})$.

$$y - 2\sqrt{3} = y'(2)(x-2)$$

$$y'(2) = -\frac{x}{y}\Big|_{(2,2\sqrt{3})} = -\frac{2}{2\sqrt{3}} = -\frac{1}{\sqrt{3}}$$

$$y - 2\sqrt{3} = -\frac{1}{\sqrt{3}}(x-2)$$

EXAMPLE 3. *Find* $\frac{dy}{dx}$ *for the following:*

(a) $4x^3 + 2y^2 = 4xy^5 + y$

$$\left(4x^3 + 2\left[y(x)\right]^2\right)' = \left(4x\left[y(x)\right]^5 + y(x)\right)'$$

$$12x^2 + 4y\,y' = 4y^5 + 20xy^4y' + y'$$

$$4yy' - 20xy^4y' - y' = -12x^2 + 4y^5$$

$$y'\left[4y - 20xy^4 - 1\right] = -12x^2 + 4y^5$$

$$y' = \frac{-12x^2 + 4y^5}{4y - 20xy^4 - 1}$$

(b) $x^3 - \cot(xy^2) = x\cos y$

$$\frac{d}{dx}\left(x^3 - \cot\left(x[y(x)]^2\right)\right) = \frac{d}{dx}\left(x\cos(y(x))\right)$$

$$3x^2 + \csc^2(xy^2)\cdot\frac{d}{dx}\left(x[y(x)]^2\right) = \cos y - xy'\sin y$$

$$3x^2 + \csc^2(xy^2)\left[y^2 + \underline{x\cdot 2yy'}\right] = \cos y - \underline{xy'\sin y}$$

$$2xy\csc^2(xy^2)\,y' + xy'\sin y = \cos y - 3x^2 - y^2\csc^2(xy^2)$$

$$xy'\left(2\csc^2(xy^2) + \sin y\right) = \cos y - 3x^2 - y^2\csc^2(xy^2)$$

$$y' = \frac{\cos y - 3x^2 - y^2\csc^2(xy^2)}{x\left(2\csc^2(xy^2) + \sin y\right)}$$

(c) $(x^2 + y^2)^5 = x^2 y^3$

$$\left(x^2 + [y(x)]^2\right)^5 = x^2 [y(x)]^3$$

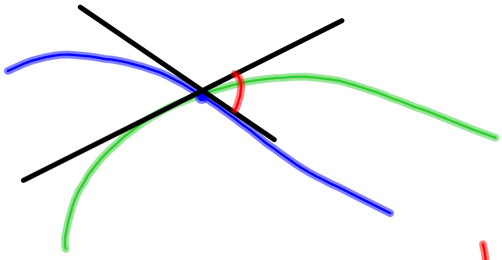
$$5(x^2+y^2)^4 \cdot (2x + 2yy') = 2xy^3 + 3x^2y^2y'$$

$$10x(x^2+y^2)^4 + 10y(x^2+y^2)^4 y' = 2xy^3 + 3x^2y^2y'$$

$$y'\left(10y(x^2+y^2)^4 - 3x^2y^2\right) = 2xy^3 - 10x(x^2+y^2)^4$$

$$y' = \frac{2xy^3 - 10x(x^2+y^2)^4}{10y(x^2+y^2)^4 - 3x^2y^2}$$

DEFINITION 4. *Two curves are said to be* **orthogonal** *if at the point(s) of their intersection, their tangent lines are orthogonal(perpendicular). In this case we also say that the angle between these curves is* $\frac{\pi}{2}$.



$0 < \text{angle} < \pi$

1) $y = m_1 x + b \quad (L_1)$

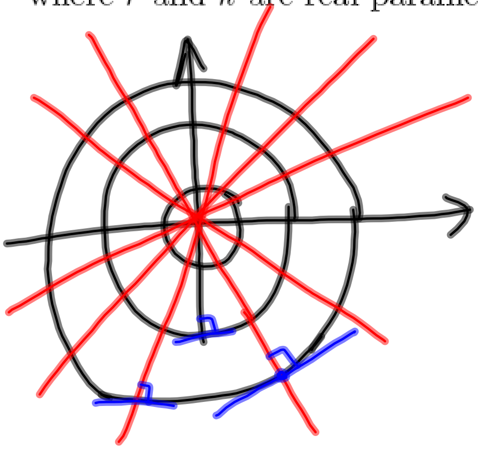
$y = m_2 x + b \quad (L_2)$

$L_1 \perp L_2 \iff m_1 \cdot m_2 = -1$

`Illustration:` Consider two families of curves:

$$x^2 + y^2 = r^2, \quad y = kx,$$

where r and k are real parameters.



$$\Downarrow$$

$$x^2 + [y(x)]^2 = r^2$$

$$\frac{d}{dx}\left(x^2 + (y(x))^2\right) = \frac{d}{dx}(r^2)$$

$$2x + 2yy' = 0$$

$$yy' = -x$$

$$m_1 = y' = -\frac{x}{y}$$

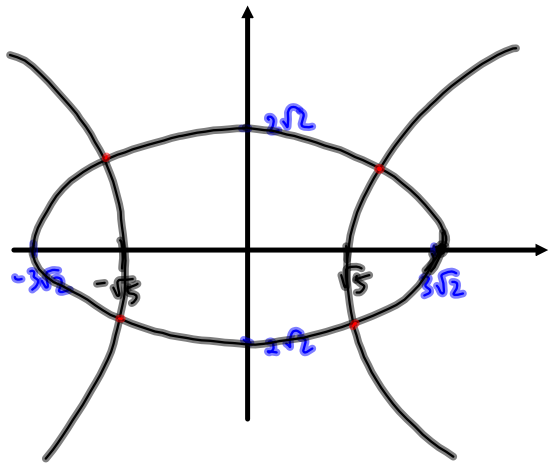
Tangent line slope to $x^2+y^2=r^2$ at (x,y)

slope $m_2 = k = \frac{y}{x}$

$$m_1 \cdot m_2 = -\frac{x}{y} \cdot \left(\frac{y}{x}\right) = -1$$

The curves are orthogonal (perpendicular)

EXAMPLE 5. *Are these curves orthogonal?*



$$x^2 - y^2 = 5, \quad 4x^2 + 9y^2 = 72$$

$x=0 \Rightarrow 9y^2 = 72$

$y^2 = 8$

$y = \pm 2\sqrt{2} \approx 2.8$

$y=0 \quad 4x^2 = 72$

$x^2 = 18 = 2 \cdot 9$

$x = \pm 3\sqrt{2} \approx 4.2$

Find intersection points

$x^2 - y^2 = 5 \Rightarrow x^2 = 5 + y^2$

$72 = 4x^2 + 9y^2 = 4(5+y^2) + 9y^2 = 20 + 13y^2$

$52 = 13y^2$

$y^2 = \frac{52}{13} = 4 \quad \Rightarrow y = \pm 2$

$x = \pm\sqrt{5+y^2} = \pm 3$

Find tangent to

$x^2 - y^2 = 5$ $\qquad$ $4x^2 + 9y^2 = 72$

at $x = \pm 3$ $\quad (y = \pm 2)$

Use implicit differentiation:

$2x - 2yy' = 0$

$8x + 18yy' = 0$

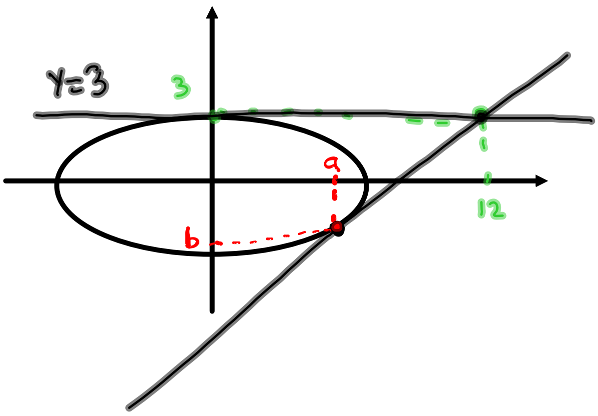
$m_1 = y' = \frac{x}{y}$

$m_2 = y' = -\frac{8x}{18y} = -\frac{4x}{9y}$

$$m_1 \cdot m_2 = \frac{x}{y} \cdot \left(-\frac{4x}{9y}\right) = -\frac{4}{9}\frac{x^2}{y^2} \overset{\substack{x=\pm 3 \\ y=\pm 2}}{=\!=} -\frac{4}{9} \cdot \frac{9}{4} = -1$$

YES: At $(3,2), (-3,2), (3,-2), (-3,-2)$ the curves are orthogonal

EXAMPLE 6. *Find the equations of both the tangent lines to the ellipse $x^2+4y^2=36$ that pass through the point* $(12,3)$.



Slope = ? (use Implicit Diff.)

$$x^2 + 4\left[y(x)\right]^2 = 36$$

$$2x + 8yy' = 0$$

$$y' = -\frac{2x}{8y} = -\frac{x}{4y}$$

$$m = y'\Big|_{(x,y)=(a,b)} = -\frac{a}{4b}$$

Tangent line to ellipse at (a,b) with slope $m = -\frac{a}{4b}$

$$y - b = -\frac{a}{4b}(x-a)$$

(12,3) belongs to the tangent

$$\Rightarrow \begin{cases} 3-b = -\frac{a}{4b}(12-a) \\ a^2+4b^2=36 \end{cases} \quad (*)$$

(a,b) belongs to the ellipse $\Rightarrow$ $a^2+4b^2=36$

$$3-b=-\frac{a}{4b}(12-a) \qquad (\times 4b)$$

$$12b-4b^2=-12a+a^2$$

$$12b+12a=a^2+4b^2\overset{(*)}{=}36$$

$$12(a+b)=36 \Rightarrow \boxed{a+b=3}$$

Plug in it to $(*)$

$$\Downarrow$$

$$b=3-a$$

$$a^2+4(3-a)^2=36$$

$$a(5a-24)=0$$

check this step

$a=0$ or $a=\frac{24}{5}$

$$b=3-a=3-\frac{24}{5}=-\frac{9}{5}$$

$$m=-\frac{a}{4b}=-\frac{\cancel{24}^{\cancel{6}\,2}}{\cancel{4}\cdot\cancel{5}}\cdot\left(-\frac{\cancel{5}}{\cancel{9}_3}\right)=\frac{2}{3}$$

$y=3$

and

$y+\frac{9}{5}=\frac{2}{3}\left(x-\frac{24}{5}\right)$