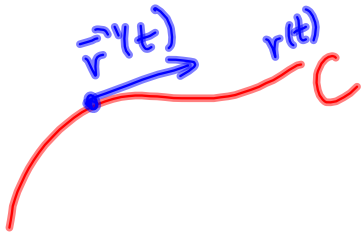


3.9: Slopes and tangents of parametric curves

Consider a curve C given by the parametric equations

$$x = x(t), \quad y = y(t),$$

or in vector form:

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath}$$

If both $x(t)$ and $y(t)$ are differentiable, then

$$\text{velocity} \quad \mathbf{r}'(t) = \langle x'(t), y'(t) \rangle$$

is a vector that is tangent to C. Its slope is:

$$\boxed{\text{slope} = \frac{y'(t)}{x'(t)}}$$

Another way to see this is by using the Chain Rule. We have $y = y(x(t))$ and then
$\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{\mathrm{d}y}{\mathrm{d}x}\dfrac{\mathrm{d}x}{\mathrm{d}t}$ which implies $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}$. $= \dfrac{y'(t)}{x'(t)} = \text{slope}$

EXAMPLE 1. *Find the equation of tangent line to the curve*

$$x(t) = \sin t, \quad y(t) = \tan t$$

at the point corresponding to $t = \dfrac{\pi}{4}$.

Tangent point $\left(x\left(\frac{\pi}{4}\right), y\left(\frac{\pi}{4}\right)\right) = \left(\sin\frac{\pi}{4}, \tan\frac{\pi}{4}\right) = \left(\frac{\sqrt{2}}{2}, 1\right)$

slope $m = \dfrac{y'(t)}{x'(t)} = \dfrac{\sec^2 t}{\cos t} = \dfrac{1}{\cos^3 t}$

$$m\left(\frac{\pi}{4}\right) = \frac{1}{\cos^3\left(\frac{\pi}{4}\right)} = \frac{1}{\left(1/\sqrt{2}\right)^3} = (\sqrt{2})^3 = 2\sqrt{2}$$

$$y - y_0 = m(x - x_0)$$

$$\boxed{y - 1 = 2\sqrt{2}\left(x - \frac{\sqrt{2}}{2}\right)}$$

Note: in parametric form:

$x = \frac{\sqrt{2}}{2} + x'\left(\frac{\pi}{4}\right)t$

$y = 1 + y'\left(\frac{\pi}{4}\right)t$

$\vdots$

EXAMPLE 2. *Find the equation of tangent line to the curve*

$$x(t) = t + 1, \quad y(t) = t^2 + 4$$

at the $(2, 5)$.

Find t s.t. $\langle x(t), y(t) \rangle = \langle 2, 5 \rangle$

$t + 1 = 2 \Rightarrow t = 1$

$t^2 + 4 = 5 \Rightarrow t = \pm 1$

$m = \frac{y'}{x'} = \frac{2t}{1} \Rightarrow m(1) = \frac{2 \cdot 1}{1} = 2$

$\boxed{y - 5 = 2(x - 2)}$

EXAMPLE 3. *Find the points on the curve*

$$x = t + t^2, \quad y = t^2 - t$$

where the tangent lines are horizontal and there they are vertical.

horizontal

$m = 0$

$\frac{y'(t)}{x'(t)} = 0$

$\Downarrow$

$y'(t) = 0$ but $x'(t) \neq 0$

$2t - 1 = 0$ but $1 + 2t \neq 0$

$t = \frac{1}{2}$ $\quad t \neq -\frac{1}{2}$

$t = \frac{1}{2}$

$\left(x\left(\frac{1}{2}\right), y\left(\frac{1}{2}\right)\right) = \boxed{\left(\frac{3}{4}, -\frac{1}{4}\right)}$

tangent horizontal

$x'(t) = 1 + 2t$

$y'(t) = 2t - 1$

$m = \frac{y'(t)}{x'(t)}$

vertical

$m =$ undefined

$x'(t) = 0$ but $y'(t) \neq 0$

$t = -\frac{1}{2}$ $\quad t \neq \frac{1}{2}$

$\left(x\left(-\frac{1}{2}\right), y\left(-\frac{1}{2}\right)\right) = \boxed{\left(-\frac{1}{4}, \frac{3}{4}\right)}$

vertical tange

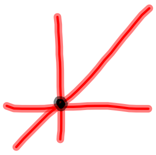
Note It may happen that $x'(t) = y'(t) = 0$ for some value of t.

Ex. 1 $x = t^3, \ y = t^3$

$x'(0) = y'(0) = 0$

What about tangent?

$x = y$



We can reparameterize the given curve s.t. $x'(0) \neq 0$ and $y'(0) \neq 0$ simultaneously.

F.ex. $x = t, \ y = t$

$x'(t) = 1, \quad y'(t) = 1$

$m = \frac{1}{1} = 1$

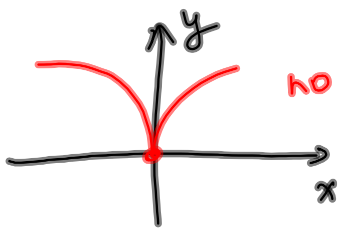
Ex. 2 $x = t^3, \ y = t^2$

$x'(0) = y'(0) = 0$

Eliminate parameter

$x^2 = t^6, \quad y^3 = t^6$

$x^2 = y^3$



no tangent at $t=0$.

EXAMPLE 4. *Show that the curve*

$$x = \cos t, \quad y = \cos t \sin t = \frac{1}{2}\sin 2t$$

has two tangents at $(0, 0)$ *and find their equations.*

$$(0,0) = \left(\cos t, \frac{1}{2}\sin 2t\right)$$

$$\begin{cases} \cos t = 0 \implies t = \frac{\pi}{2}, \frac{3\pi}{2} \\ \frac{1}{2}\sin 2t = 0 \implies 2t = 0, \pi, 2\pi, 3\pi \end{cases}$$

$$t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$$

$$m = \frac{y'}{x'} = \frac{\cos 2t}{-\sin t}$$

$$m\left(\frac{\pi}{2}\right) = -\frac{\cos \pi}{\sin \frac{\pi}{2}} = 1$$

$$m\left(\frac{3\pi}{2}\right) = -\frac{\cos(3\pi)}{\sin \frac{3\pi}{2}} = -\frac{-1}{-1} = -1$$

$$y = mx$$

$$\boxed{y = x}$$

$$\boxed{y = -x}$$