## 4.4:Derivatives of Logarithmic Functions

EXAMPLE 1. *Using Implicit Differentiation find the derivatives of the following function:*

**(a)** $f(x) = \ln x$

$$y = \ln x$$

$$x = e^{y(x)}$$

$$\frac{d}{dx}(x) = \frac{d}{dx}\left(e^{y(x)}\right)$$

$$1 = y'(x)\, e^{y(x)} \Rightarrow y'(x) = \frac{1}{e^{y(x)}} = \frac{1}{x}$$

$$\boxed{(\ln x)' = \frac{1}{x}}$$

(b) $f(x) = a^x$

$$y = a^x$$

$$\ln y = \ln a^x$$

$$\frac{d}{dx}\left(\ln y(x)\right) = \frac{d}{dx}(x \ln a)$$

$$\frac{1}{y} y' = \ln a \Rightarrow y' = y \ln a$$

$$y' = a^x \ln a$$

$$\boxed{(a^x)' = a^x \ln a}$$

*Combining* the formulas obtained in Example 1 and Chain Rule one can get

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln(g(x)) = \frac{1}{g(x)}\,g'(x)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}a^{g(x)} = a^{g(x)}\ln a \cdot g'(x)$$

EXAMPLE 2. *Find the derivative:*

(a) $f(x) = \overbrace{\ln}^{\text{outer}}\underbrace{(\sin x)}_{\text{inner function}}$

$$f'(x) = \frac{1}{\sin x}(\sin x)' = \frac{\cos x}{\sin x} = \cot x$$

(b) $f(x) = \overbrace{\ln}^{\text{outer}}(\overbrace{1 + \ln(1 + \ln x)}^{\text{inner}})$

$$f'(x) = \frac{1}{1+\ln(1+\ln x)} \cdot (1 + \overbrace{\ln}^{\text{outer}}\overbrace{(1+\ln x)}^{\text{inner}})'$$

$$= \frac{0 + \frac{(1+\ln x)'}{1+\ln x}}{1+\ln(1+\ln x)} = \frac{\frac{1}{x}}{(1+\ln x)(1+\ln(1+\ln x))}$$

(c) $f(x) = \ln|x| = \begin{cases} \ln x, & x > 0 \\ \ln(-x), & x < 0 \end{cases}$

$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x \leq 0 \end{cases}$

$$f'(x) = \begin{cases} (\ln x)', & x > 0 \\ (\ln(-x))', & x < 0 \end{cases} = \begin{cases} \frac{1}{x}, & x > 0 \\ \frac{1}{-x} \cdot (-1) = \frac{1}{x}, & x < 0 \end{cases}$$

$$(\ln|x|)' = \frac{1}{x}$$

(d) $f(x) = \cot^3 x + 3^{\cot x} = [\underbrace{\cot x}_{\text{inner}}]^3 + 3^{\cot x}$ ← inner function

outer $[\ \ ]^3$

$3^{[\ \ ]}$ outer

$(3^a)' = 3^a \ln 3$

$$f'(x) = \left[3\cot^2 x + 3^{\cot x} \ln 3\right](-\csc^2 x)$$

EXAMPLE 3. *Using the change of base formula, find the derivative formula for* $f(x) = \log_a x$ *and generalize it using the Chain Rule.*

$$\frac{d}{dx}\left[\log_a x\right] = \left(\frac{\ln x}{\ln a}\right)' = \frac{1}{\ln a}\,(\ln x)' = \frac{1}{x\ln a}$$

$$\boxed{(\log_a x)' = \frac{1}{x\ln a}}$$

Generalize $\left[\log_a \underbrace{g(x)}_{\text{inner function}}\right]' = \frac{1}{g(x)\ln a}\,g'(x)$

EXAMPLE 4. *Find the derivative of* $f(x) = \log_2(\underbrace{3 + x^2 + x^3}_{\text{inner}})$

$$f'(x) = \frac{1}{(3+x^2+x^3)\ln 2}\cdot(3+x^2+x^3)'$$

$$f'(x) = \frac{2x+3x^2}{(3+x^2+x^3)\ln 2}$$

Logarithmic Differentiation can be used to find derivative of complicated functions involving products, quotients or powers.

*STEPS IN LOGARITHMIC DIFFERENTIATION:*

1. Take logarithms of both sides of an equation $y = f(x)$ and simplify (f.ex. split a product or quotient, etc.).
2. Differentiate implicitly with respect to $x$.
3. Solve the resulting equation for $y'$.
4. Plug in $y = f(x)$.

$\boxed{\text{Due Wednesday}}$

<u>Ex</u>

$$f(x) = \frac{e^{x^2}(x^5+5)^5}{\sqrt[3]{x^3+3} \cdot \sin^2 x}$$

$$[\ln f(x)]' = \frac{f'(x)}{f(x)}$$

EXAMPLE 5. *Find the derivative:* $y = (\cos x)^{\sin x}$

$$\ln y = \ln (\cos x)^{\sin x}$$

$\ln a^b = b \ln a$

$$\ln y(x) = \sin x \ln(\underbrace{\cos x}_{\text{inner}})$$

$\downarrow$ P.R.

$$\frac{y'(x)}{y(x)} = \cos x \ln(\cos x) + \sin x \frac{(-\sin x)}{\cos x}$$

$$y'(x) = y(x)\left[\cos x \ln(\cos x) - \frac{\sin^2 x}{\cos x}\right]$$

$$\boxed{y'(x) = (\cos x)^{\sin x}\left[\cos x \ln(\cos x) - \frac{\sin^2 x}{\cos x}\right]}$$