4.5: Exponential Growth and Decay

If $y(t)$ is the value of a quantity y at time t and if the rate of change of y with respect to t is proportional to its size $y(t)$ at any time, then

$$\underbrace{\frac{\mathrm{d}y}{\mathrm{d}t} = ky}_{\text{differential equation}} \iff y' = ky$$

where k is a constant.

THEOREM 1. *The only solution of the* differential equation $\frac{\mathrm{d}y}{\mathrm{d}t} = ky$ *are the exponential functions*

$$\boxed{y(t) = y_0 e^{kt}}$$

where $y_0 = y(0)$ *is the initial quantity.*

In the problems below your primary goal is to find the constant k.

EXAMPLE 2. *A bacteria culture starts with 4000 bacteria and the population triples every half-hour.*

(a) *Find an expression for the number of bacteria after t hours.* $= y(t)$

$y(0) = 4000$

$y\left(\frac{1}{2}\right) = 3\,y(0) = 12000$

$\frac{dy}{dx} = ky \Rightarrow \boxed{y = y(0)e^{kt} = 4000\,e^{kt}}$

To find k use

$y\left(\frac{1}{2}\right) = \cancel{4000}\,e^{\frac{k}{2}} = \cancel{12000}\,3$

$(e^{k})^{\frac{1}{2}} = 3$

$e^{k} = 9$

$y(t) = 4000\,(e^{k})^{t}$

$\boxed{y(t) = 4000 \cdot 9^{t}}$, t in hours

(b) *Find the number of bacteria after 20 minutes.* $= \frac{1}{3}h$

$y\left(\frac{1}{3}\right) = 4000 \cdot 9^{\frac{1}{3}} \approx \ldots$

EXAMPLE 3. *After 3 days a sample of radon-222 decayed to 58% of its original amount.* $y(0)$

(a) *What is the half-life of radon-222?* $y(t) = \frac{1}{2} y(0)$ for what t

$y(t)$ amount of radon-222 after t days

(*) $y(3) = 0.58\, y(0)$

Exp. decay $y(t) = y(0) e^{kt}$

To determine k use (*).

$$\cancel{y(0)} e^{3k} = 0.58\, \cancel{y(0)}$$

$$(e^k)^3 = 0.58 \Rightarrow e^k = 0.58^{\frac{1}{3}}$$

Plug in $y(t) = y(0) (e^k)^t = y(0) \cdot 0.58^{t/3}$

We have to find t such that

$$y(t) = \frac{1}{2} y(0)$$

$$\cancel{y(0)} \cdot 0.58^{t/3} = \frac{1}{2} \cancel{y(0)}$$

$$\ln 0.58^{\frac{t}{3}} = \ln \frac{1}{2}$$

$$\frac{t}{3} \ln 0.58 = \ln 0.5$$

$$t = \frac{3 \ln 0.5}{\ln 0.58} \approx 3.82 \text{ days}$$

(b) *How long would it take the sample to decay to 10% of its original amount?*

Find t s.t.

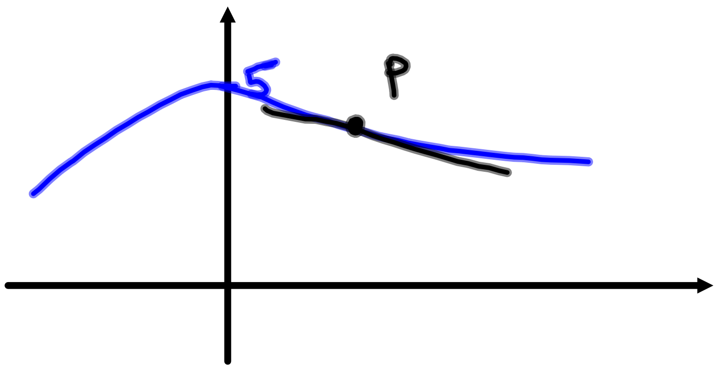
$$y(t) = 0.1\, y(0)$$

$$\cancel{y(0)} \cdot 0.58^{\frac{t}{3}} = 0.1\, \cancel{y(0)}$$

$$\frac{t}{3} \ln 0.58 = \ln 0.1$$

$$t = \frac{3 \ln 0.1}{\ln 0.58} \simeq 12.68 \text{ days}$$

EXAMPLE 4. *A curve passes through the point $(0, 5)$ and has the property that the slope of the curve at every point P is twice the y-coordinate of P. Find the equation of the curve.*



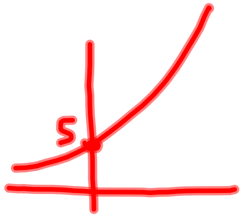
slope $\begin{cases} y'(x) = 2y \\ y(0) = 5 \end{cases}$ $\Rightarrow K = 2$

We know that the solution of this initial value problem is

$$y(x) = y(0)e^{Kx}$$

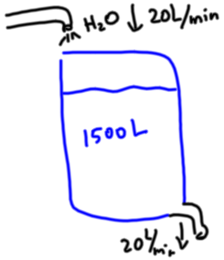
or

$$\boxed{y(x) = 5e^{2x}}$$



EXAMPLE 4. *A tank contains 1500L of brine with a concentration of* 0.3 kg *of salt per liter. In order to dilute the solution, pure water is run into the tank at a rate of* 20L/min. *The resulting solution, which is kept mixed, runs out at the same rate.*

(a) *How many kilograms of salt will remain after half an hour?* $= 30\text{min}.$



Concentration of salt $= 0.3 \, kg/L$

$y(t)$ amount of salt at tank at time t (in min.)

Find $y(30) = ?$

$$\frac{dy(t)}{dt} = \begin{pmatrix}\text{rate in}\\ \text{of salt}\end{pmatrix}(t) - \begin{pmatrix}\text{rate out}\\ \text{of salt}\end{pmatrix}(t)$$

$$\text{rate in}(t) = \begin{pmatrix}\text{concentration}\\ \text{of salt} \quad (t)\end{pmatrix} \cdot \underbrace{\begin{pmatrix}\text{rate}\\ \text{in}\end{pmatrix}}_{=0} = 0$$

$$\text{rate out}(t) = \begin{pmatrix}\text{concentration}\\ \text{of salt} \quad (t)\end{pmatrix} \cdot \begin{pmatrix}\text{rate}\\ \text{out}\end{pmatrix} = \frac{y(t)}{\cancel{1500}_{75}} \cdot \cancel{20}^{1}$$

$$\frac{dy}{dt} = 0 - \frac{y(t)}{75} \Rightarrow \boxed{\frac{dy}{dt} = -\frac{1}{75} y} \Rightarrow k = -\frac{1}{75}$$

$$y(t) = y(0) e^{-t/75}$$

$y(0)$ amount of salt in tank at the beginning.

$$y(0) = 0.3 \frac{kg}{L} \cdot 1500 L = 450 \, kg.$$

$$\boxed{y(t) = 450 \, e^{-t/75}}$$

After half an our

$$y(30) = 450 e^{-\frac{30}{75}} \approx 301.6 \, kg$$

(b) *When will the concentration be reduced to* 0.2 kg *of salt per liter?*

$$\Rightarrow y(t) = 0.2 \frac{kg}{L} \cdot 1500 L = 300 \, kg$$

Find t such that $y(t) = 300$

$$\cancel{450}^{3} e^{-t/75} = \cancel{300}^{2}$$

$$e^{-t/75} = \frac{2}{3}$$

$$-\frac{t}{75} \underbrace{\ln e}_{=1} = \ln \frac{2}{3}$$

$$t = -75 \ln \frac{2}{3} \approx 30.41 \text{ min}$$

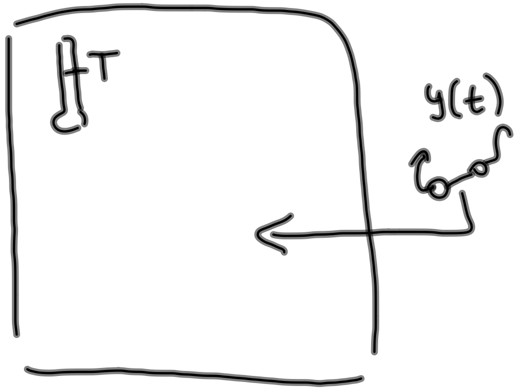
Newton's Law of Cooling states that *the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings.* For example, if $y(t)$ is the temperature of the object at time t and T is the the temperature of the room in which the object is cooling then

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}t} = k(y - T).}$$

The solution of this equation, which gives the temperature of the object at time t, is

$$\boxed{y(t) = (y_0 - T)e^{kt} + T,}$$

where $y_0 = y(0)$ is the initial temperature of the object. Indeed,



Denote $\boxed{u = y - T}$

$$\frac{du}{dt} = \frac{dy}{dt} - 0 = k(y-T) = ku$$

We have

$$\frac{du}{dt} = ku \Rightarrow u = u(0)e^{kt}$$

$$y - T = (y(0) - T)e^{kt}$$

$$\boxed{y = (y(0) - T)e^{kt} + T}$$

EXAMPLE 6. *A thermometer is taken from a room where the temperature is 20°C to the outdoors, where the temperature is 5°C. After one minute, the temperature reads 12°C.*

(a) *What will the reading of the thermometer be after one more minute?*

$T = 5°$

$y(0) = 20°$

$\boxed{y(1) = 12°}$

$y(2) = ?$

$y(t)$ = temperature of the thermometer at time t (minutes)

Using Newton's Law of Cooling we get

$$y(t) = (y(0) - T)\,e^{kt} + T$$

Use this condition to determine k:

$$12 = y(1) = (20-5)\,e^{k\cdot 1} + 5$$

$$7 = 15e^{k} \Rightarrow \boxed{e^{k} = \frac{7}{15}}$$

$$e^{kt} = (e^{k})^{t}$$

$$y(t) = (20-5)\left(\frac{7}{15}\right)^{t} + 5$$

$$\boxed{y(t) = 15\cdot\left(\frac{7}{15}\right)^{t} + 5} \Rightarrow y(2) = \cancel{15}\cdot\frac{7^2}{15^{\cancel{2}}} + 5 = \frac{49 + 5\cdot 15}{15}$$

$$y(2) \approx 8.27°C$$

(b) *When will the thermometer read 6°C?*

Find t such that $y(t) = 6$

$$6 = 15\cdot\left(\frac{7}{15}\right)^{t} + 5$$

$$\frac{1}{15} = \left(\frac{7}{15}\right)^{t} \Rightarrow \ln\frac{1}{15} = t\ln\frac{7}{15}$$

$$t = \frac{\ln\frac{1}{15}}{\ln\frac{7}{15}} \approx 3.55 \text{ min}$$