4.8:Indeterminate forms and L'Hospital's Rule

Indeterminate forms: Consider

$$\lim_{x\to a}\frac{f(x)}{g(x)}. \tag{1}$$

- If both $f(x)\to 0$ and $g(x)\to 0$ as $x\to a$, then (1) is called an **indeterminate form of type** $\frac{0}{0}$.
- If both $f(x)\to\pm\infty$ and $g(x)\to\pm\infty$ as $x\to a$, then (1) is called an **indeterminate form of type** $\frac{\infty}{\infty}$.

EXAMPLES:

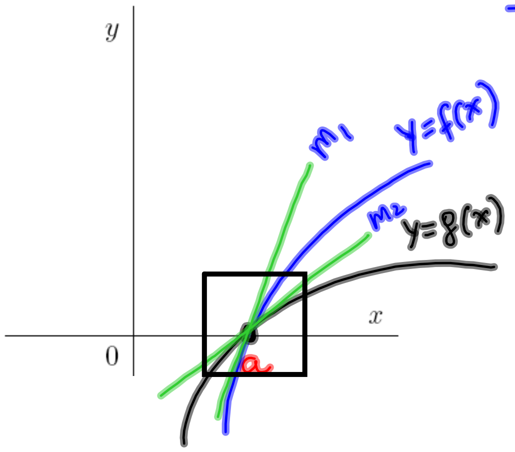
$$\lim_{x\to 0}\frac{\sin x}{x}=\frac{0}{0},\quad \lim_{x\to 1}\frac{x-x^2}{x^2-1}=\frac{0}{0},\quad \lim_{x\to 0}\frac{3^x-1}{x^2}=\frac{0}{0},\quad \lim_{x\to\infty}\frac{\ln x}{x^3}=\frac{\infty}{\infty},\quad \lim_{x\to\infty}\frac{x^2+1}{4x^2-1}=\frac{\infty}{\infty}.$$

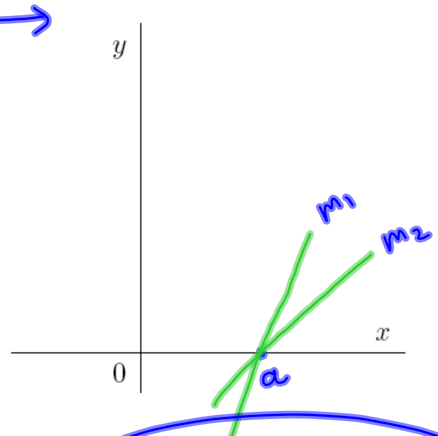
L'HOSPITAL'S RULE: Suppose f and g are differentiable and $\boxed{\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}}$ or $\frac{\infty}{\infty}$ then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

Zoom ++ ...





$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}\Rightarrow \begin{matrix} f(a)=0\\ g(a)=0\end{matrix}$$

near $x=a$:

$$\frac{f(x)}{g(x)}\approx\frac{\text{Linearization } f(x)}{\text{Linearization } g(x)}$$

$$\frac{f(x)}{g(x)}\approx\frac{\cancel{f(a)}^{0}+f'(a)\cancel{(x-a)}}{\cancel{g(a)}^{0}+g'(a)\cancel{(x-a)}}$$

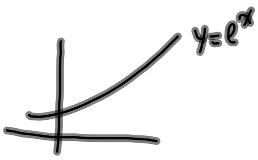
$$\frac{f(x)}{g(x)}\approx\frac{f'(a)}{g'(a)}$$

EXAMPLE 1. *Evaluate each of the following limits:*

(a) $\lim\limits_{x\to 0}\dfrac{\sin x}{x} \overset{\frac{0}{0}}{=} \lim\limits_{x\to 0}\dfrac{\cos x}{1} = \cos 0 = 1$

(b) $\lim\limits_{x\to\infty}\dfrac{x^2}{e^x} = \dfrac{\infty}{\infty} \overset{H}{=} \lim\limits_{x\to\infty}\dfrac{(x^2)'}{e^x} = \lim\limits_{x\to\infty}\dfrac{2x}{e^x} \underset{H}{\overset{\frac{\infty}{\infty}}{=}}$

$$= \lim_{x\to\infty}\frac{2}{e^x} = 0$$



(c) $\displaystyle \lim_{x\to 0} \frac{\sin x - x}{x^3} \overset{0/0}{\underset{H}{=}} \lim_{x\to 0} \frac{\cos x - 1}{3x^2} \overset{0/0}{\underset{H}{=}}$

$$= \lim_{x\to 0} \frac{-\sin x}{6x} \overset{0/0}{\underset{H}{=}} \lim_{x\to 0} -\frac{\cos x}{6} = \boxed{-\frac{1}{6}}$$

(d) $\displaystyle \lim_{x\to\infty} \frac{(\ln x)^5}{x^4} \overset{\infty/\infty}{\underset{H}{=}} \lim_{x\to\infty} \frac{5(\ln x)^4}{x \cdot 4x^3} \overset{\infty/\infty}{\underset{H}{=}}$

$$= \lim_{x\to\infty} \frac{5}{4} \frac{4(\ln x)^3}{x \cdot 4x^3}$$ = Differentiate 3 more times (do it!) to get $\displaystyle \lim_{x\to\infty} \frac{\text{constant}}{4x^4} = 0$

Indeterminate form of type $0 \cdot \infty$: $\lim_{x \to a} f(x)g(x)$

Write the product fg as a quotient to get an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$:

$f(x) \to 0$

$g(x) \underset{x \to a}{\to} \infty$

$$fg = \frac{f}{\frac{1}{g}} = \frac{0}{0}$$

OR

$$fg = \frac{g}{1/f} = \frac{\infty}{\infty}$$

EXAMPLE 2. *Evaluate each of the following limits:*

(a) $\lim_{x\to 0^+} x^2 \ln x \overset{0\cdot\infty}{=} \lim_{x\to 0^+} \frac{\ln x}{x^{-2}} \underset{H}{\overset{\frac{\infty}{\infty}}{=}} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-2x^{-3}} = -\frac{1}{2}\lim_{x\to 0^+} \frac{x^3}{x}$

$= -\frac{1}{2}\lim_{x\to 0^+} x^2 = \boxed{0}$

(b) $\lim_{x\to -\infty} xe^x = \infty \cdot 0$

$\lim_{x\to -\infty} \frac{e^x}{x^{-1}} \underset{H}{\overset{\frac{0}{0}}{=}} \lim_{x\to \infty} \frac{e^x}{-x^{-2}} = -\lim x^2 e^x \to$ STOP

OR

$\lim_{x\to -\infty} \frac{x}{e^{-x}} \underset{H}{\overset{\frac{\infty}{\infty}}{=}} \lim_{x\to -\infty} \frac{1}{-e^{-x}} = 0$

Indeterminate form of type $\infty - \infty$: $\lim\limits_{x\to a}(f(x) - g(x))$

Try to convert the difference into a quotient to get an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

→ common denominator

→ factorize

→ rationalize (multiply by conjugate)

→ common factor

EXAMPLE 3. *Find:* $\lim\limits_{x\to 1}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$

$$= \lim_{x\to 1} \frac{x-1-\ln x}{\ln x\,(x-1)} \quad \overset{\frac{0}{0}}{\underset{H}{=}} \quad \text{differentiate}$$

$$= \lim_{x\to 1} \frac{1-\frac{1}{x}}{\frac{1}{x}(x-1)+\ln x} = \lim_{x\to 1} \frac{x-1}{x-1+x\ln x} \quad \overset{0/0}{\underset{H}{=}}$$

$$= \lim_{x\to 1} \frac{1}{1+\ln x + x\cdot\frac{1}{x}} = \lim_{x\to 1}\frac{1}{2+\ln x} = \frac{1}{2+0} = \boxed{\frac{1}{2}}$$

Indeterminate form of type 0^0, ∞^0, 1^∞: $\lim_{x\to a} f(x)^{g(x)}$

Write the function as an exponential $0 \cdot \infty$:

$$f^g = e^{\ln f^g} = e^{g \ln f}$$

$$\boxed{\begin{aligned} a &= e^{\ln a} \\ x &= e^{\ln x} \end{aligned}}$$

It leads to an indeterminate form of type $0 \cdot \infty$.

$$\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} e^{g \ln f} = e^{\lim_{x\to a} g \ln f(x)}$$

EXAMPLE 4. *Find the following limits:*

(a) $\lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} e^{\ln x^{\frac{1}{x}}} = \lim_{x\to\infty} e^{\frac{1}{x}\ln x} = e^{\lim_{x\to\infty}\frac{\ln x}{x}} = e^{0} = \boxed{1}$

$\lim_{x\to\infty} \frac{\ln x}{x} \overset{\frac{\infty}{\infty}}{\underset{H}{=}} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$

(b) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} e^{x\ln\left(1+\frac{1}{x}\right)} = e^{\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right)} = e^1 = \boxed{e}$

$$\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) \overset{\infty\cdot 0}{=\!=} \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \overset{\frac{0}{0}}{\underset{H}{=}} \lim_{x\to\infty}\frac{\Big(\ln\big(\overbrace{1+\tfrac{1}{x}}^{\text{inner fun.}}\big)\Big)'}{\left(\frac{1}{x}\right)'}$$

$$= \lim_{x\to\infty}\frac{\frac{1}{1+\frac{1}{x}}\cdot\cancel{\left(-\frac{1}{x^2}\right)}}{\cancel{-\frac{1}{x^2}}} = \lim_{x\to\infty}\frac{1}{1+\frac{1}{x}} = 1$$

(c) $\lim_{x\to0+}(\sin x)^{\tan x} = e^{\lim_{x\to0^+}\tan x\,\ln(\sin x)} = e^0 = \boxed{1}$ Final answer

$$\lim_{x\to0^+}\tan x\,\ln(\sin x) \overset{0\cdot\infty}{=\!=} \lim_{x\to0^+}\frac{\ln(\sin x)}{\frac{1}{\tan x}} = \lim_{x\to0^+}\frac{\overbrace{\ln}^{\text{outer}}\overbrace{(\sin x)}^{\text{inner}}}{\cot x} \overset{\frac{\infty}{\infty}}{\underset{H}{=}}$$

$$= \lim_{x\to0^+}\frac{\frac{1}{\sin x}\cos x}{-\csc^2 x} = -\lim_{x\to0^+}\frac{\cos x}{\cancel{\sin x}}\cdot\sin^{\cancel{2}} x = -\lim_{x\to0^+}\cos x\sin x =$$

$$= -\cos 0\cdot\sin 0 = \boxed{0}$$