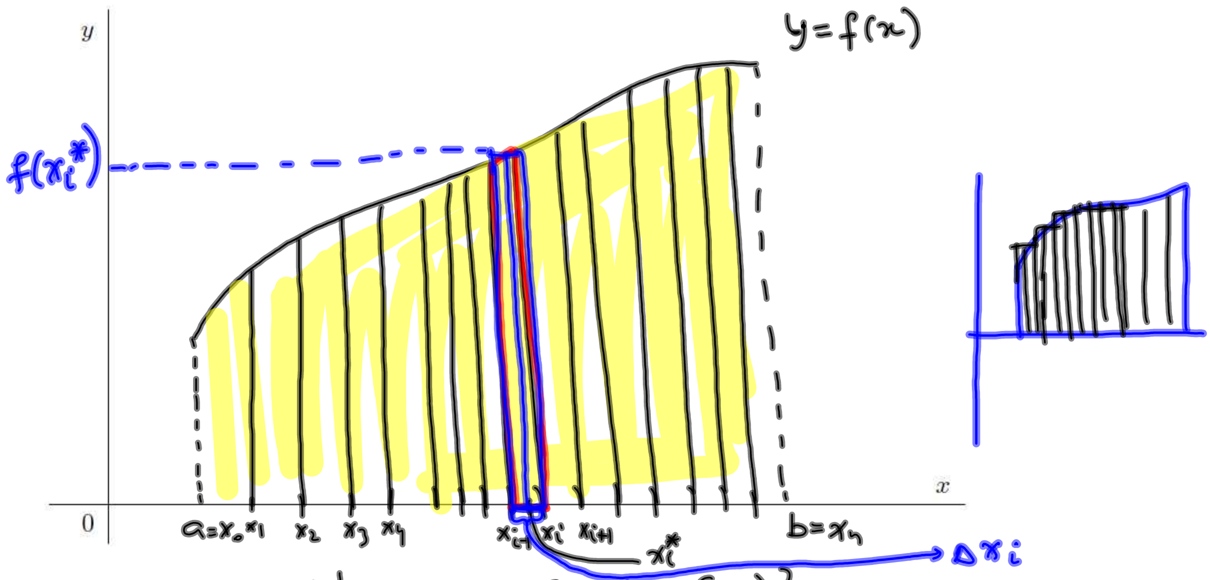
6.2: Area

Area problem: Let a function $f(x)$ be positive on some interval $[a, b]$. Determine the area of the region between the function and the x-axis.



$D = \{(x,y) \mid a \le x \le b,\ 0 \le y \le f(x)\}$

Solution: Choose **partition** points $x_0, x_2, \ldots, x_{n-1}, x_n$ so that

$$a = x_0 \le x_2 \le \ldots \le x_{n-1} \le x_n = b.$$

Base of rectangle

Use notation $\Delta x_i = x_i - x_{i-1}$ for the length of ith subinterval $[x_{i-1}, x_i]$ $(1 \le 1 \le n)$
The length of the longest subinterval is denoted by $\|P\|$. $= \max \Delta x_i$ norm of partition
The location in each subinterval where we compute the height is denoted by x_i^*.
The area of the ith rectangle is

$$A_i = f(x_i^*)\,\Delta x_i$$

Then

$$A \approx \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$

The area A of the region is:

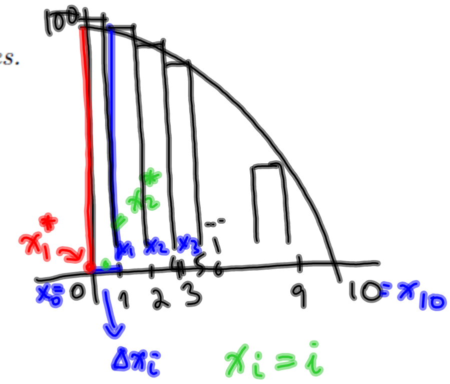
$$A = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

EXAMPLE 1. *Given* $f(x) = 100 - x^2$ *on* $[0, 10]$. *Let* $P = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ *and* x_i^* *be left endpoint of* i*th subinterval.*

(a) *Find* $\|P\|$.

(b) *Find the sum of the areas of the approximating rectangles.*

(c) *Sketch the graph of* f *and the approximating rectangles.*



(a)

$\Delta x_i = 1$ for all i $\Rightarrow \|P\| = \max_i |\Delta x_i| = 1$

(b)

$$A \approx \sum_{i=1}^{10} f(x_i^*)\Delta x_i = \sum_{i=1}^{10} f(i-1)\cdot 1$$

$$= \sum_{i=1}^{10} \left[100 - (i-1)^2\right] = \sum_{i=1}^{10} \left[100 - i^2 + 2i - 1\right]$$

$$= \sum_{i=1}^{10} 99 - \sum_{i=1}^{10} i^2 + 2\sum_{i=1}^{10} i = 99 \cdot 10 - \frac{\cancel{10}^{5} \cdot 11 \cdot \cancel{21}^{7}}{\cancel{6}\,\cancel{3}} + \cancel{2} \cdot \frac{10 \cdot 11}{\cancel{2}}$$

$= 990 - 55 \cdot 7 + 110$

$= 715$

Note Real area ≈ 666.66

$n = 10$: $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

$n = 10$: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Riemann Sum for a function $f(x)$ on the interval $[a,b]$ is a sum of the form:

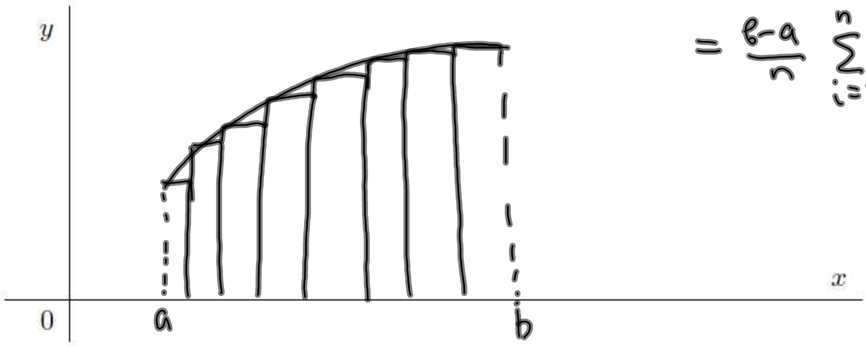
$$\boxed{\sum_{i=1}^{n} f(x_i^*)\Delta x_i.} \qquad \frac{b-a}{n}\sum_{i=1}^{n} f(x_i^*)$$

Consider a partition has equal subintervals: $x_i = a + i\Delta x$, where $\Delta x = \dfrac{b-a}{n}$.

$x_i^* = x_{i-1}$

LEFT-HAND RIEMANN SUM : $\quad L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x = \sum_{i=1}^{n} f(a + (i-1)\Delta x)\Delta x$

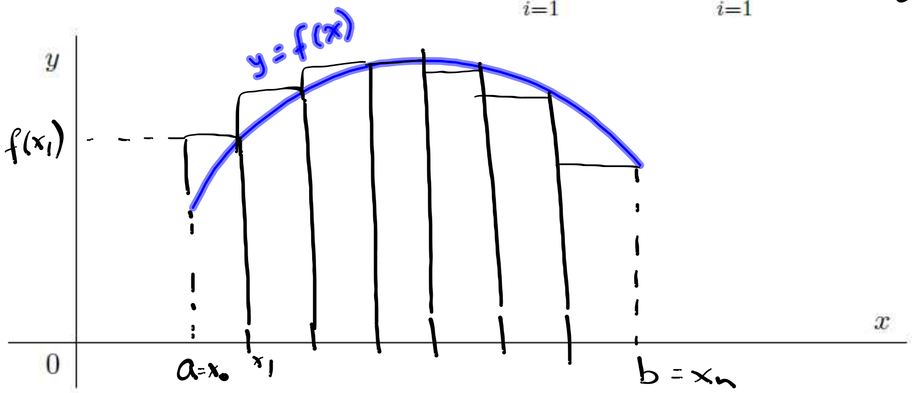
$$= \frac{b-a}{n}\sum_{i=1}^{n} f\left(a + (i-1)\frac{b-a}{n}\right)$$



$x_i^* = x_i$

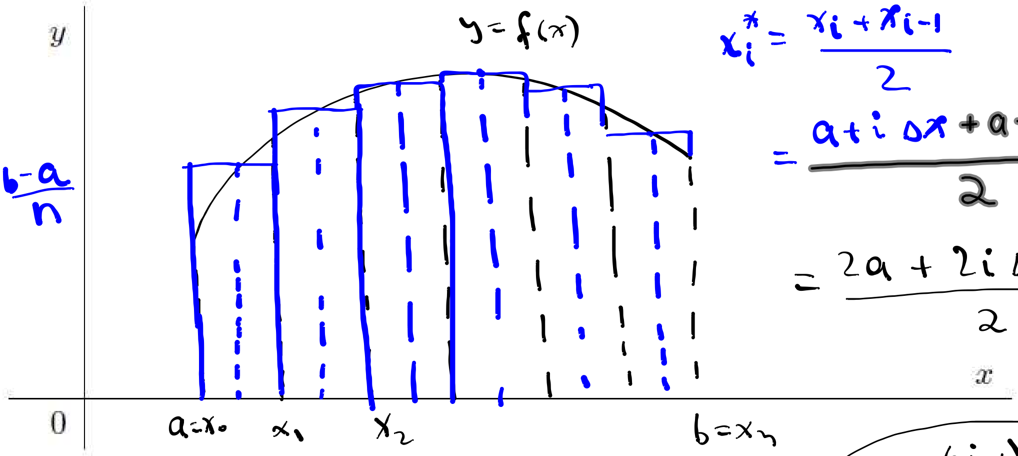
RIGHT-HAND RIEMANN SUM : $R_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} f(a+i\Delta x)\Delta x$

$\Delta x = \frac{b-a}{n}$



MIDPOINT RIEMANN SUM : $M_n = \sum_{i=1}^{n} f\left(\frac{x_i + x_{i-1}}{2}\right)\Delta x =$

$\Delta x = \frac{b-a}{n}$



$$x_i^* = \frac{x_i + x_{i-1}}{2}$$

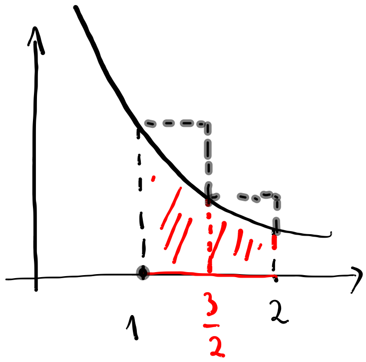
$$= \frac{a + i\Delta x + a + \overbrace{(i-1)\Delta x}^{i\Delta x - \Delta x}}{2}$$

$$= \frac{2a + 2i\Delta x - \Delta x}{2}$$

$$= a + \frac{(2i-1)\Delta x}{2}$$

$n=2, \; a=1, \; b=2$

EXAMPLE 2. *Given* $f(x) = \dfrac{1}{x}$ *on* $[1,2]$. *Calculate* L_2, R_2, M_2.

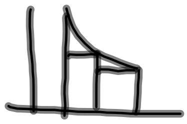


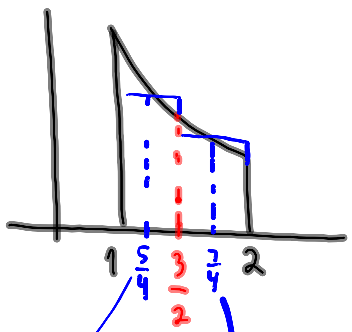
$$\Delta x = \frac{b-a}{2} = \frac{2-1}{2} = \frac{1}{2}$$

$$L_2 = \frac{1}{2}\left[f(1) + f\left(\tfrac{3}{2}\right)\right] = \frac{1}{2}\left[1 + \frac{1}{3/2}\right] = \frac{1}{2}\left[1 + \frac{2}{3}\right]$$

$$L_2 = \frac{5}{6} \approx \boxed{0.833}$$

$$R_2 = \frac{1}{2}\left[f\left(\tfrac{3}{2}\right) + f(2)\right] = \frac{1}{2}\left[\frac{2}{3} + \frac{1}{2}\right] = \frac{7}{12} \approx \boxed{0.583}$$





$$M_2 = \frac{1}{2}\left[f\left(\tfrac{5}{4}\right) + f\left(\tfrac{7}{4}\right)\right]$$

$$M_2 = \frac{1}{2}\left[\frac{4}{5} + \frac{4}{7}\right] = \frac{24}{35} \approx \boxed{0.686}$$

$$\frac{1+\frac{3}{2}}{2} = \frac{5}{4} \qquad \frac{\frac{3}{2}+2}{2} = \frac{7}{4}$$

Actual area ≈ 0.693

$$A = \lim_{\|P\| \to 0} \sum_{i=1}^{\infty} f(x_i^*)\Delta x = \underbrace{\lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{\infty} f(x_i^*)}$$ Riemann Sum

EXAMPLE 3. *Represent area bounded by $f(x)$ on the given interval using Riemann sum. Do not evaluate the limit.*

(a) $f(x) = x^2 + 2$ *on* $[0, 3]$ *using right endpoints.*

$f(x) = x^2 + 2 \geqslant 0$

$a = 0$, $b = 3$

$x_i^* = x_i = a + i\,\Delta x = 0 + i\frac{3}{n} = \frac{3i}{n}$

$$A = \lim_{n\to\infty} \overbrace{\frac{3-0}{n}}^{=\Delta x} \sum_{i=1}^{n} \left[(x_i^*)^2 + 2 \right]$$

$$A = \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^2 + 2 \right]$$

(b) $f(x) = \sqrt{x^2+2}$ *on* $[0,3]$ *using left endpoints.*

$$f(x) = \sqrt{x^2+2} > 0$$

$$\left.\begin{array}{l} a=0 \\ b=3 \end{array}\right\} \Rightarrow \Delta x = \frac{b-a}{n} = \frac{3}{n}$$

$$x_i^* = x_{i-1} = a + (i-1)\Delta x = 0 + (i-1)\cdot\frac{3}{n} = \frac{3(i-1)}{n}$$

$$f(x_i^*) = \sqrt{\left[\frac{3(i-1)}{n}\right]^2 + 2}$$

$$A = \lim_{n\to\infty} \Delta x \sum_{i=1}^{n} f(x_i^*) = \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \sqrt{\left[\frac{3(i-1)}{n}\right]^2 + 2}$$

EXAMPLE 4. *The following limits represent the area under the graph of $f(x)$ on an interval $[a, b]$. Find $f(x), a, b$.*

(a) $\lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \sqrt{1 + \frac{3i}{n}} = A = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f(x_i)$

Right-hand Rule

$\Delta x = \frac{b-a}{n} = \frac{3}{n}$

$\sqrt{1 + \frac{3i}{n}} = \sqrt{1 + i\Delta x}$

$\to 1 + i\Delta x = x_i$

$a + i\Delta x$

$a = 0 \Rightarrow b = 3$

$f(x) = \sqrt{1+x}$

$a = 1 \Rightarrow b = 4$

$f(x) = \sqrt{x}$

(b) $\displaystyle\lim_{n\to\infty} \frac{10}{n} \sum_{i=1}^{n} \frac{1}{1+\left(7+\frac{10i}{n}\right)^3} = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f(x_i^*)$

Right hand sum

$\Delta x = \frac{b-a}{n} = \frac{10}{n} \Rightarrow b-a=10$

$$\left.\begin{array}{l} x_i^* = \underbrace{x_i}_{=\,a+i\Delta x} = 7 + \frac{10i}{n} = 7 + i\Delta x \end{array}\right\} \Rightarrow a=7$$

$$\Downarrow$$

$$b = a+10 = 17$$

$$f(x) = \frac{1}{1+x^3} > 0$$

The limit represents area under $y = \frac{1}{1+x^3}$ on the interval $[7; 17]$.