6.4: The fundamental Theorem of Calculus

The fundamental Theorem of Calculus :

PART I If $f(x)$ is continuous on $[a,b]$ then $g(x) = \int_a^x f(t)\,dt$ is continuous on $[a,b]$ and differentiable on (a,b) and $g'(x) = f(x)$.

$$g(x) = \int_4^x \underbrace{\sin t^3}_{f(t)}\,dt \Rightarrow g'(x) = \sin x^3$$

EXAMPLE 1. *Differentiate* $g(x)$ *if*

(a) $$g(x) = \int_{-4}^{\boxed{x^3} = u(x)} \underbrace{e^{2t}\cos^2(1-5t)}_{f(t)}\,dt = \int_{-4}^{u(x)} f(t)\,dt = G(u(x))$$

$$g'(x) = \frac{d}{dx}\left(G(u(x))\right) \overset{\text{chain Rule}}{=} \frac{dG}{du}\cdot\frac{du}{dx} =$$

$$(x^3)' = 3x^2$$

$$= G'(u)\cdot u'(x) = \boxed{f(\overset{x^3}{u})\cdot u'(x)} =$$

$$= e^{2x^3}\cos^2(1-5x^3)\cdot 3x^2$$

(b) $g(x) = \int_{e^{x^2}}^{1} \frac{t+1}{\ln t + 3}\,\mathrm{d}t = -\int_{1}^{e^{x^2}} \underbrace{\frac{t+1}{\ln t+3}}_{f(t)}\,dt \Rightarrow$

$$\Rightarrow g'(x) = -f\left(e^{x^2}\right)\cdot\left(e^{x^2}\right)' =$$

$$= -\frac{e^{x^2}+1}{\ln e^{x^2}+3}\cdot e^{x^2}\cdot 2x =$$

$$= -\frac{2xe^{x^2}\left(e^{x^2}+1\right)}{x^2+3}$$

(c) $g(x) = \int_{x^2}^{\sin x} \frac{\cos t}{t}\,\mathrm{d}t = \int_{x^2}^{c}\frac{\cos t}{t}\,dt + \int_{c}^{\sin x}\frac{\cos t}{t}\,dt =$

$$= -\int_{c}^{x^2}\underbrace{\frac{\cos t}{t}}_{f(t)}\,dt + \int_{c}^{\sin x}\underbrace{\frac{\cos t}{t}}_{f(t)}\,dt$$

$$g'(x) = -\underbrace{\frac{\cos x^2}{x^2}}_{f(x^2)}\cdot\underbrace{2x}_{(x^2)'} + \underbrace{\frac{\cos(\sin x)}{\sin x}}_{f(\sin x)}\cdot\underbrace{\cos x}_{(\sin x)'}$$

PART II If $f(x)$ is continuous on $[a,b]$ and $F(x)$ is any antiderivative fort $f(x)$ then

$$\int_a^b f(x)\,\mathrm{d}x = F(x)\,\big|_a^b = F(b) - F(a).$$

Newton- Leibniz formula

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

DEFINITION 2. *A function F is called an* **antiderivative** *of f on an interval I if $\underline{F'(x) = f(x)}$ for all x in I.*

Table of Antidifferentiation Formulas

Function	Particular antiderivative	Most general antiderivative
$k \quad (k \in \mathbb{R})$	kx	$kx + C$
$x^n \quad (n \neq -1)$	$\frac{x^{n+1}}{n+1}$	$\frac{x^{n+1}}{n+1} + C$
$\frac{1}{x}$	$\ln\lvert x\rvert$	
e^x	e^x	
$\cos x$	$\sin x$	
$\sin x$	$-\cos x$	+ C
$\sec^2 x$	$\tan x$	
$\csc^2 x$	$-\cot x$	
$\sec x \tan x$	$\sec x$	
$\csc x \cot x$	$\csc x$	
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x$	
$\frac{1}{1+x^2}$	$\arctan x$	

EXAMPLE 3. *Evaluate*

(a) $\int_{\ln 2}^{\ln 10} \frac{1}{2} e^x \, dx = \frac{1}{2} \int_{\ln 2}^{\ln 10} e^x \, dx = \frac{1}{2} e^x \Big|_{\ln 2}^{\ln 10} =$

$$= \frac{1}{2}\left(e^{\ln 10} - e^{\ln 2}\right) = \frac{1}{2}(10-2) = 4$$

(b) $\int_{-1}^{5} \frac{1}{x^2} \, dx$ The integral DNE, because the function 1/x 2is not continuous at x=0. The given integral is so called improper integral and will be discussed later in Section 8.9.

(c) $\int_{1}^{2} \frac{2x^5 - x + 3}{x^2} \, dx = \int_1^2 \frac{2x^5}{x^2} - \frac{x}{x^2} + \frac{3}{x^2} \, dx =$

$$= \int_1^2 2x^3 - \frac{1}{x} + \frac{3}{x^2} \, dx =$$

$$= \overset{1}{\cancel{2}} \frac{x^4}{\underset{2}{\cancel{4}}} - \ln|x| - \frac{3}{x} \Big|_1^2$$

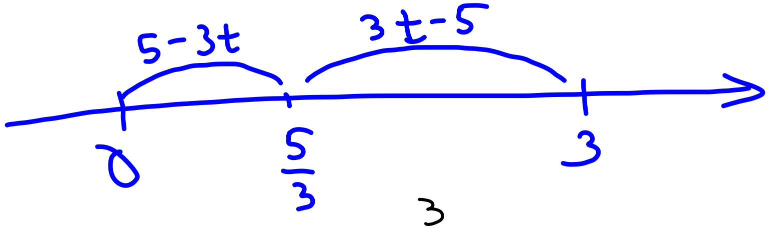
$$= \frac{2^4}{2} - \ln 2 - \frac{3}{2} - \left(\frac{1}{2} - \overset{=0}{\ln 1} - 3\right)$$

$$= 8 - \ln 2 - \frac{3}{2} - \frac{1}{2} + 3 = 9 - \ln 2$$

(d) $\int_0^3 |3t-5|\,dt$

$$|x| = \begin{cases} x, & x \geqslant 0 \\ -x, & x \leqslant 0 \end{cases}$$

$$|3t-5| = \begin{cases} 3t-5, & 3t-5 \geqslant 0 \quad 3t \geqslant 5 \quad t \geqslant \frac{5}{3} \\ -(3t-5)=5-3t, & 3t-5 \leqslant 0, \quad t \leqslant \frac{5}{3} \end{cases}$$



$$\int_0^{5/3} (5-3t)\,dt + \int_{5/3}^{3} (3t-5)\,dt =$$

$$= \left(5t - \frac{3t^2}{2}\right)\Bigg|_0^{5/3} + \left(\frac{3t^2}{2} - 5t\right)\Bigg|_{\frac{5}{3}}^{3} =$$

$$= \dots = \frac{41}{6}$$