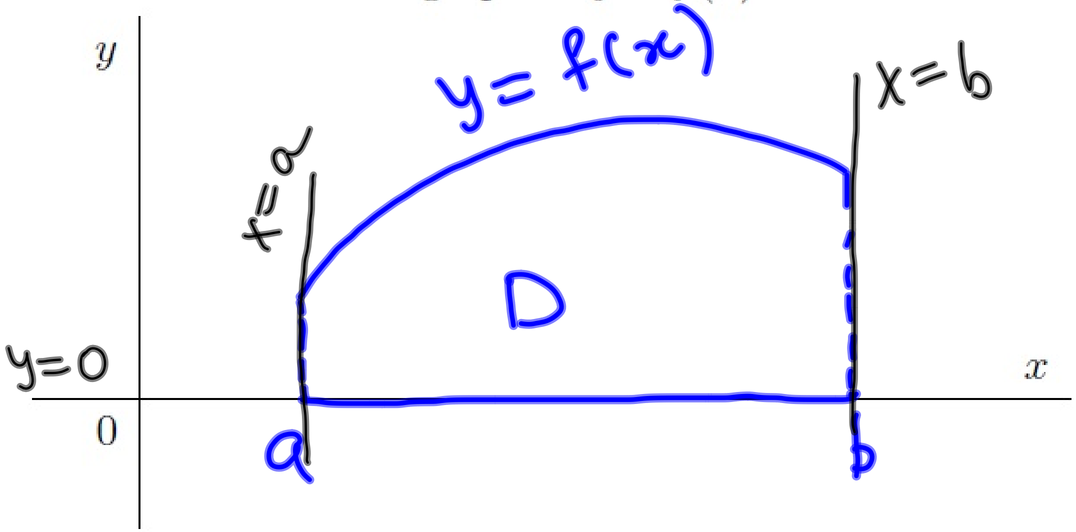
7.1: Areas Between Curves

One of interpretations of definite integral

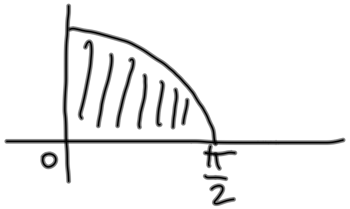
$$\text{Area}(D) = \int_a^b f(x)\,dx, \quad f(x) \geq 0 \text{ on } [a,b]$$

is the area between the graph of $y = f(x)$ and the x-axis on $[a,b]$.



$$D = \{(x,y) \mid a \leq x \leq b,\ 0 \leq y \leq f(x)\}$$

For example, if $f(x) = \cos x$ *and* $x \in [0, \frac{\pi}{2}]$ *then*

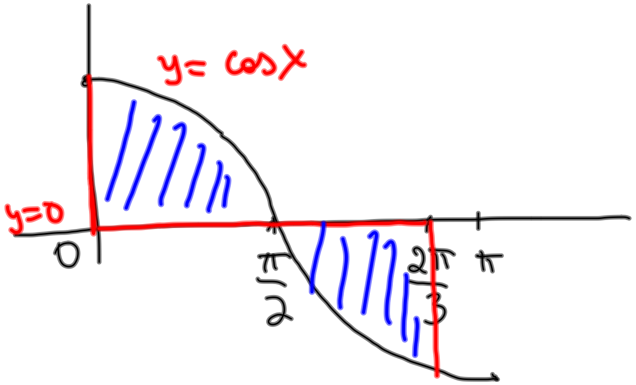


$$\text{Area} = \int_0^{\pi/2} \cos x \, dx = \sin x \Big|_0^{\frac{\pi}{2}} = 1$$

If $f(x) \geq 0$ on $[a, b]$ then $\int_a^b f(x)\, dx \geq 0$

If $f(x) \leq 0$ on $[a, b]$ then $\int_a^b f(x)\, dx \leq 0$

The previous example on $[0, \frac{2\pi}{3}]$*:*



$$\int_0^{2\pi/3} \cos x \, dx \neq \text{Area}$$

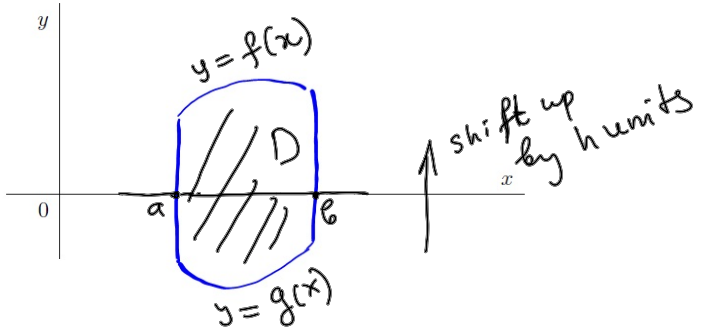
$$\int_0^{\pi/2} \cos x \, dx + \left(- \int_{\frac{\pi}{2}}^{2\pi/3} \cos x \, dx \right)$$

Our goal: **Find the area between two curves.**

CASE I. *Determine the area between $y = f(x)$ and $y = g(x)$ on the interval $[a, b]$ assuming $f(x) \geq g(x)$ on $[a, b]$.*

I other words, find the area of the region D defined by

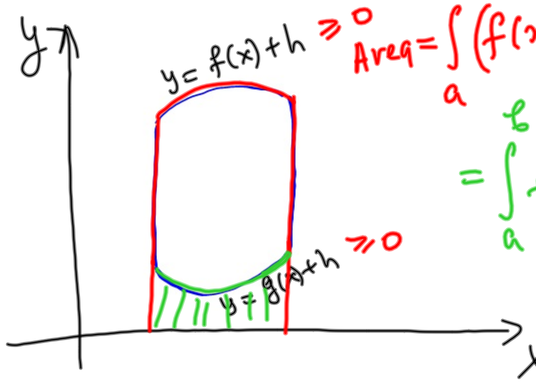
$$D = \{ (x, y) \mid a \leq x \leq b,\ g(x) \leq y \leq f(x) \}$$



Solution:

$$A = A(D) = \int_a^b f(x) - g(x)\, \mathrm{d}x$$

Explanation:



$$\text{Area} = \int_a^b (f(x) + h)\, dx - \int_a^b (g(x) + h)\, dx$$

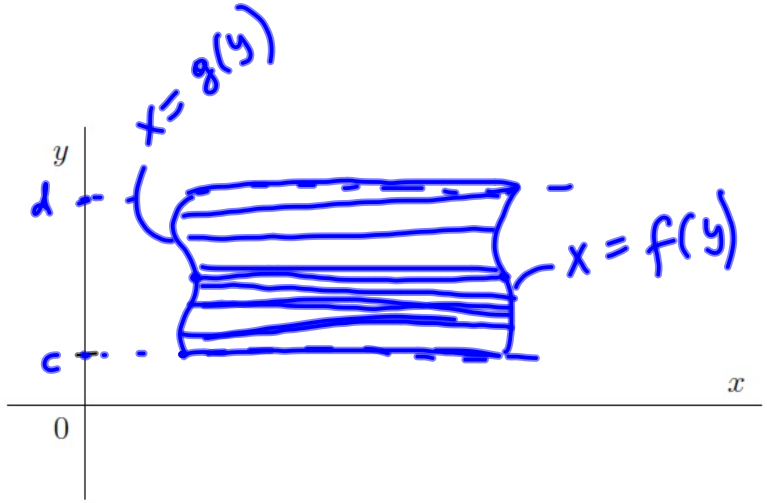
$$= \int_a^b f(x) + \cancel{h} - g(x) - \cancel{h}\ dx$$

$$= \int_a^b f(x) - g(x)\, dx =$$

$$= \int_a^b \begin{pmatrix}\text{Top}\\ \text{curve}\end{pmatrix} - \begin{pmatrix}\text{bottom}\\ \text{curve}\end{pmatrix} dx$$

CASE II. *Determine the area between* $x = f(y)$ *and* $x = g(y)$ *on the interval* $[c, d]$ *assuming* $f(y) \geq g(y)$ *on* $[c, d]$.

I other words, find the area of the region D defined by



Solution:

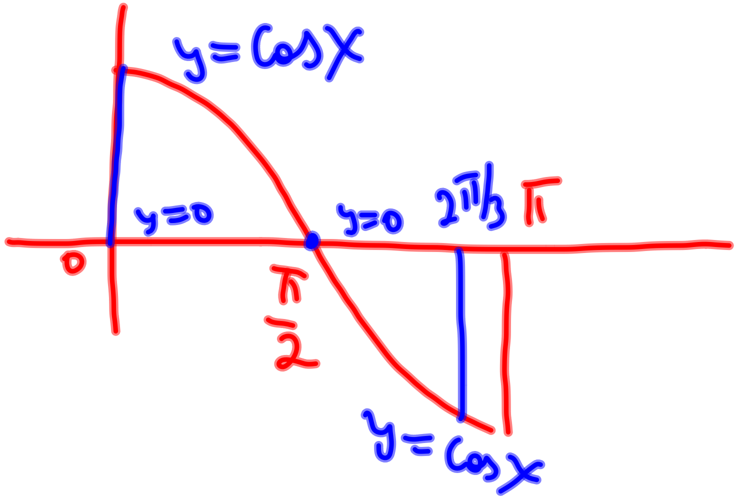
$$A = A(D) = \int_c^d f(y) - g(y)\,\mathrm{d}y$$

The above formulas in the "word" form:

$$\text{CASE I} \quad A = \int_a^b \begin{pmatrix} \text{upper} \\ \text{function} \end{pmatrix}^{(x)} - \begin{pmatrix} \text{lower} \\ \text{function} \end{pmatrix}^{(x)} \mathrm{d}x$$

$$\text{CASE II} \quad A = \int_c^d \begin{pmatrix} \text{right} \\ \text{function} \end{pmatrix}^{(y)} - \begin{pmatrix} \text{left} \\ \text{function} \end{pmatrix}^{(y)} \mathrm{d}y$$

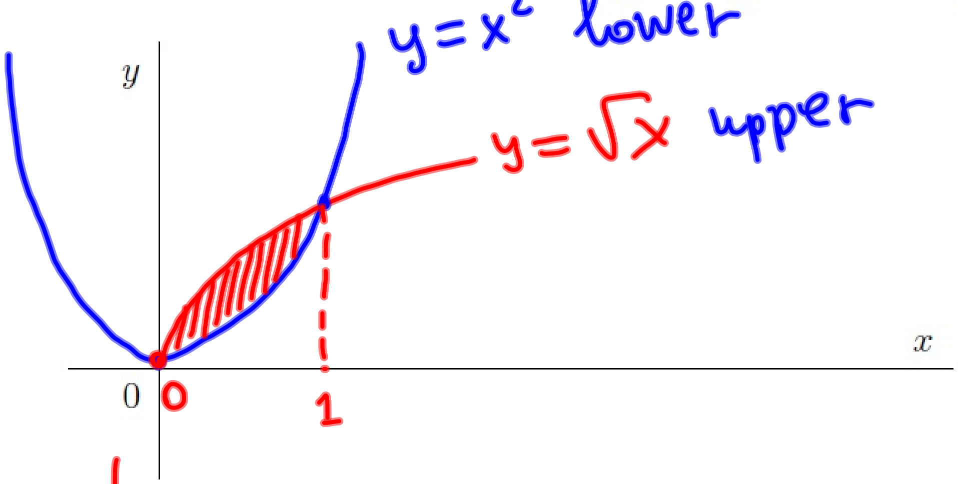
Coming back to the previous example: $f(x) = \cos x$, where $0 \leq x \leq 2\pi/3$ we get:



$$A = \int_0^{\pi/2} (\cos x - 0)\, dx + \int_{\pi/2}^{2\pi/3} (0 - \cos x)\, dx =$$

$= \ldots\ldots$

EXAMPLE 1. *Determine the area of the region enclosed (=bounded by) by $y = x^2$ and $y = \sqrt{x}$.*



$$y = x^2,\quad y = \sqrt{x} \Rightarrow x^2 = \sqrt{x}$$

$$x^4 = x$$

$$x(x^3 - 1) = 0$$

$$x = 0 \qquad x^3 = 1$$

$$x = 1$$

$$\int_0^1 (UF) - (LF)\,dx =$$

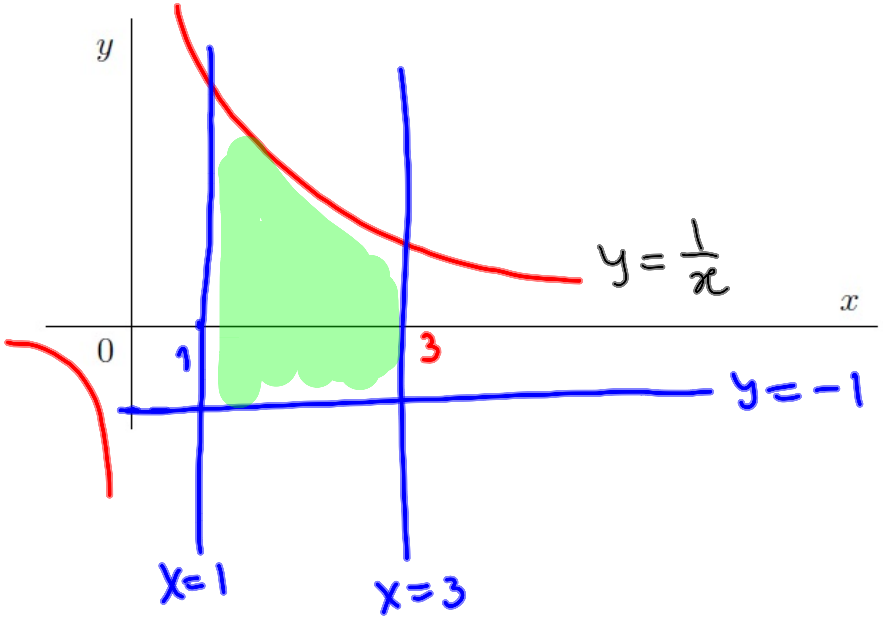
$$= \int_0^1 (\sqrt{x} - x^2)\,dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^3}{3}\Big|_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

REMARK 2. 1. The limits of integration in the above example were determined as the intersection points of the two curves.

2. Sketch of a graph of the region is recommended (it helps to determine which of the functions is upper/right).

3. The area between two curves will always be positive

EXAMPLE 3. *Determine the area of the region bounded by* $y = \frac{1}{x}$ *and* $y = -1$, $x = 1$, $x = 3$



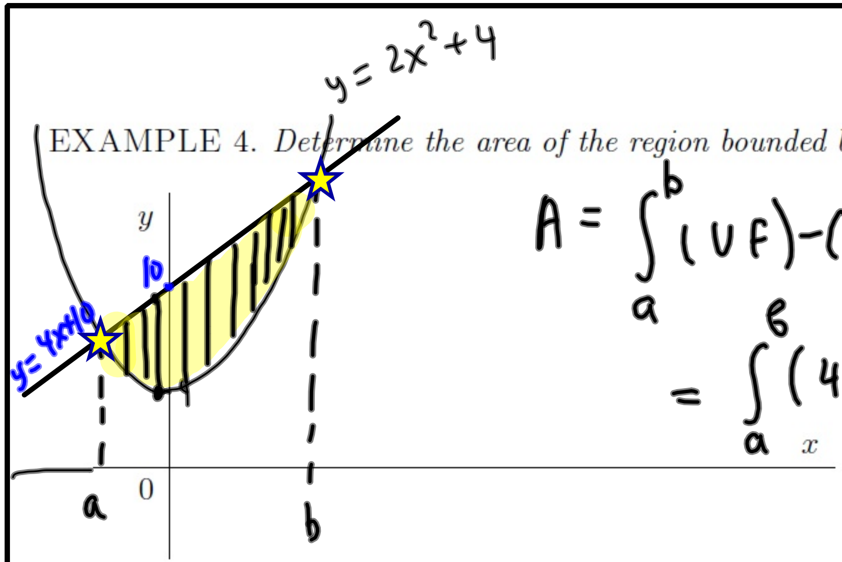
$$\text{Area} = \int_1^3 (UF) - (LF)\,dx =$$

$$= \int_1^3 \left(\frac{1}{x} - (-1)\right)dx = \ln|x| + x \Big|_1^3 =$$

$$= \ln 3 - \overset{=0}{\ln 1} + 3 - 1 =$$

$$= \ln 3 + 2$$

EXAMPLE 4. *Determine the area of the region bounded by* $y = 2x^2 + 4$ *and* $y = 4x + 10$.



$$A = \int_a^b (UF) - (LF)\,dx = \int_a^b (4x+10) - (2x^2+4)\,dx$$

$$= \int_{-1}^{3} (4x + 6 - 2x^2)\,dx = \ldots = \; = \frac{64}{3}$$

Find x-coord. of intersection points

$$\left.\begin{array}{l} y = 2x^2 + 4 \\ y = 4x + 10 \end{array}\right\} \Rightarrow 2x^2 + 4 = 4x + 10 \quad \left(\times \tfrac{1}{2}\right)$$

$$x^2 + 2 = 2x + 5$$

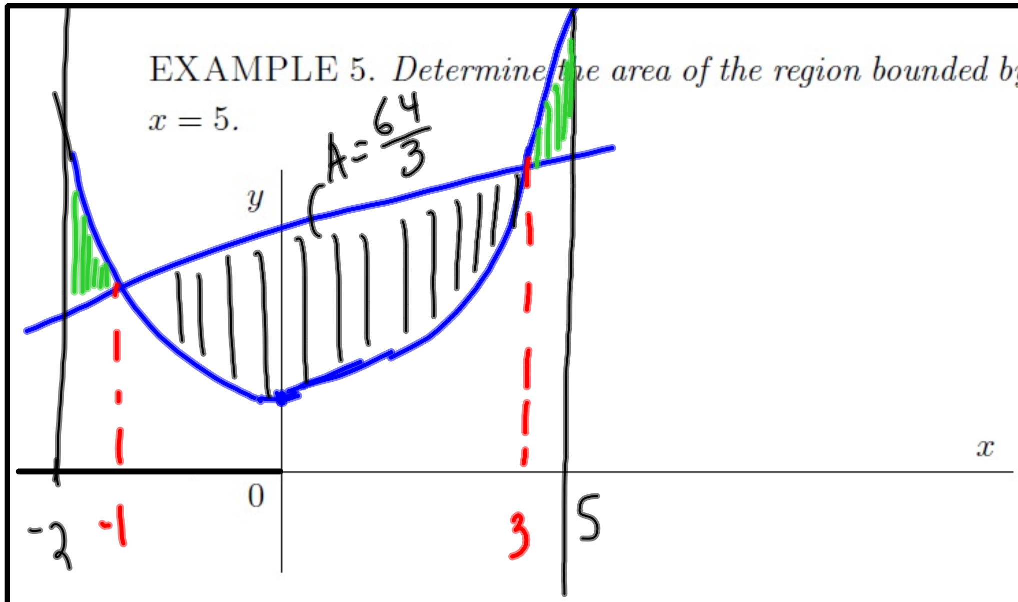
$$x^2 - 2x - 3 = 0$$

$$(x+1)(x-3) = 0$$

$$\underset{\displaystyle a}{\underset{\shortparallel}{x = -1}} \quad \text{or} \quad \underset{\displaystyle b}{\underset{\shortparallel}{x = 3}}$$

EXAMPLE 5. *Determine the area of the region bounded by* $y = 2x^2 + 4$ *and* $y = 4x + 10$, $x = -2$, $x = 5$.

as in Ex. 4

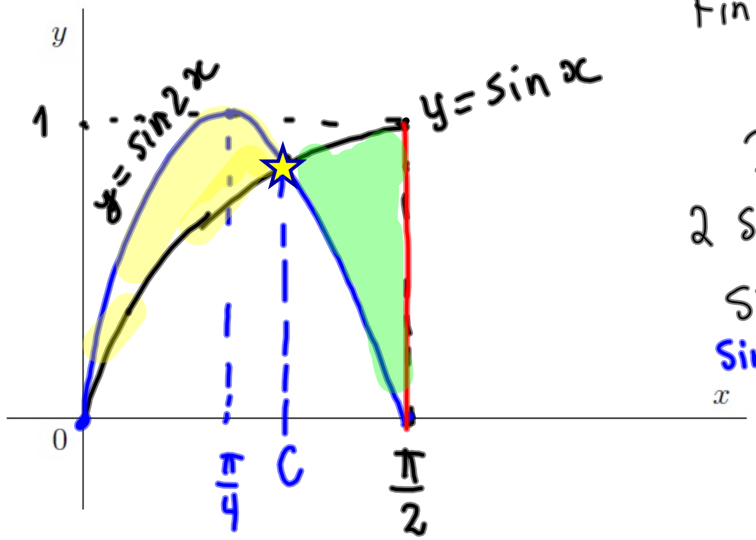


$$A = \int_{-2}^{5} (UF) - (LF)\, dx =$$

$$= \int_{-2}^{-1} 2x^2 + 4 - (4x + 10)\, dx + \frac{64}{3} + \int_{3}^{5} 2x^2 + 4 - (4x + 10)\, dx =$$

$$= \ldots = \frac{142}{3}$$

EXAMPLE 6. *Determine the area of the region enclosed by* $y = \sin x$, $y = \sin 2x$, $x = 0$, $x = \pi/2$.



Find C:

$$\sin 2x = \sin x$$

$$2 \sin x \cos x = \sin x$$

$$2 \sin x \cos x - \sin x = 0$$

$$\sin x\,(2\cos x - 1) = 0$$

$\sin x = 0$ or $2\cos x = 1$

$x = 0$ $\qquad \cos x = \frac{1}{2}$

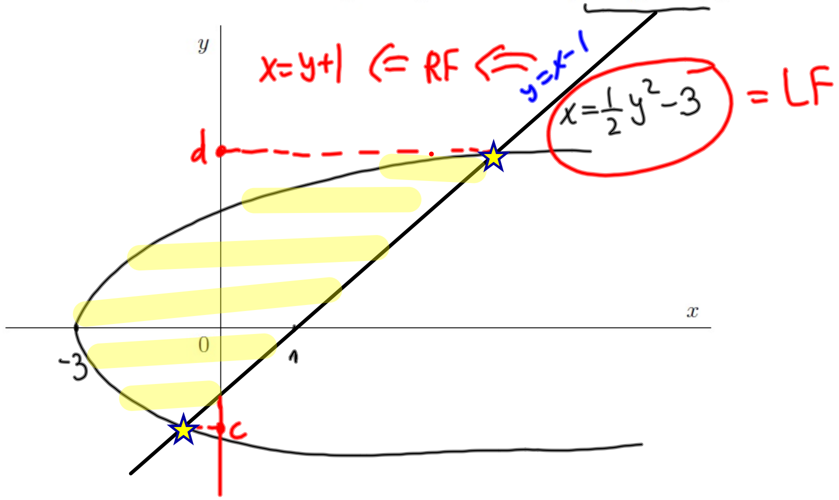
$$x = \frac{\pi}{3} = C$$

$$A = \int_0^{\pi/2} (UF) - (LF)\,dx = \int_0^{C=\frac{\pi}{3}} \sin 2x - \sin x \, dx + \int_{C=\frac{\pi}{3}}^{\pi/2} \sin x - \sin 2x \, dx =$$

$$= \left(-\frac{1}{2}\cos 2x - (-\cos x)\right)\Bigg|_0^{\pi/3} + \left(-\cos x - \left(-\frac{1}{2}\cos 2x\right)\right)\Bigg|_{\frac{\pi}{3}}^{\pi/2} =$$

$$= \dots = \frac{1}{2}$$

EXAMPLE 7. *Determine the area of the region enclosed by* $x = \frac{1}{2}y^2 - 3$, $y = x - 1$.



$$A = \int_c^d (RF)^{(y)} - (LF)^{(y)}\,dy = \int_c^d (y+1) - \left(\frac{1}{2}y^2 - 3\right) dy =$$

Find c and d:

$\frac{1}{2}y^2 - 3 = y + 1 \quad (\times 2)$

$y^2 - 6 = 2y + 2$

$y^2 - 2y - 8 = 0$

$(y-4)(y+2) = 0$

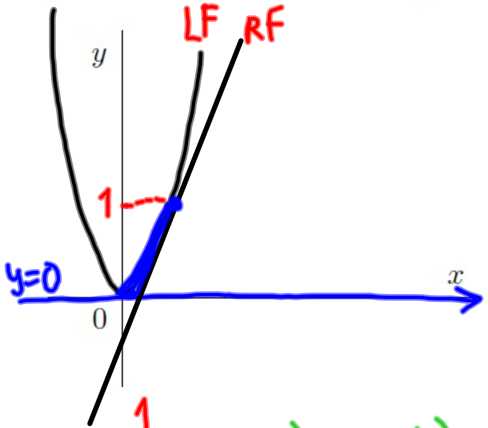
$\underset{\overset{\shortparallel}{d}}{y = 4}$ or $y = -2 = c$

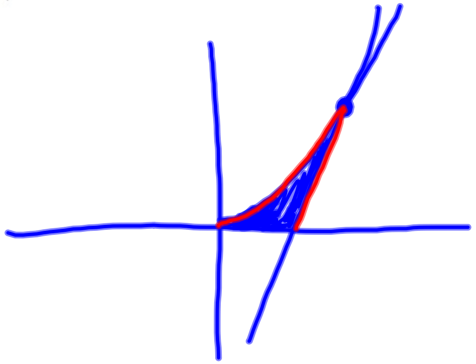
$$= \int_{-2}^{4} \left(y + 4 - \frac{1}{2}y^2\right) dy =$$

$= \ldots = 18$

EXAMPLE 8. *Determine the area of the region bounded by the x-axis, the curve* $\underbrace{y = x^2}_{}$ *and tangent line to this curve at the point* $(1,1)$.

LF





$x = \pm\sqrt{y}$

LF

$$A = \int_0^1 (RF)^{(y)} - (LF)^{(y)}\, dy =$$

$$= \int_0^1 \left(\frac{y+1}{2} - \sqrt{y}\right) dy =$$

$$= \frac{y^2}{4} + \frac{y}{2} - \frac{2}{3} y^{\frac{3}{2}} \Big|_0^1 =$$

$$= \frac{1}{4} + \frac{1}{2} - \frac{2}{3} = \boxed{\frac{1}{12}}$$

Tangent $y = mx + b$

$m = (x^2)'\Big|_{x=1} = 2$

$y = 2x + b$

$(1,1)$: $1 = 2 + b \Rightarrow b = -1$

$$\boxed{y = 2x - 1}$$

$x = \frac{y+1}{2}$ (RF)