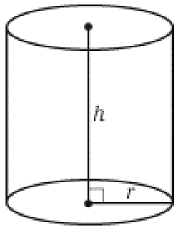
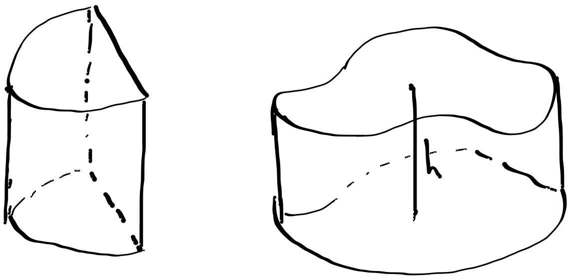
7.2: VOLUME

A simple type of solid: right **cylinder**



$V = \left(\begin{smallmatrix}\text{Base}\\ \text{Area}\end{smallmatrix}\right) \cdot (\text{Height})$

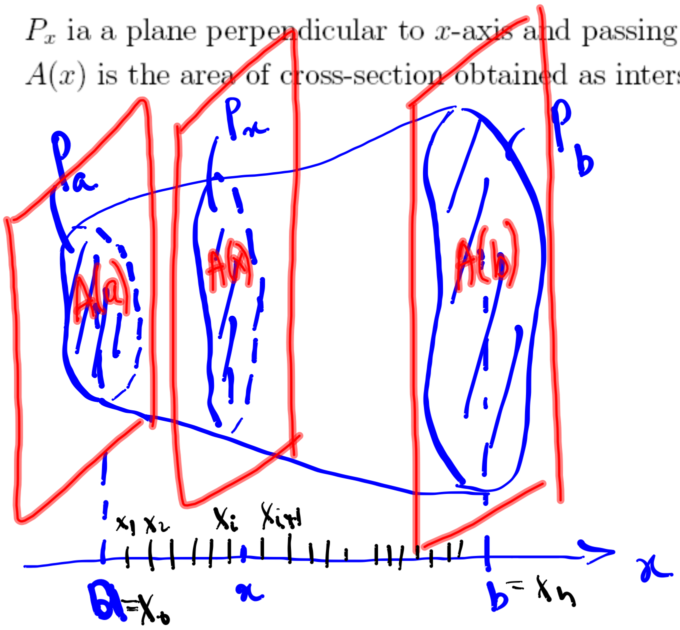
$V = \pi r^2 \cdot h$

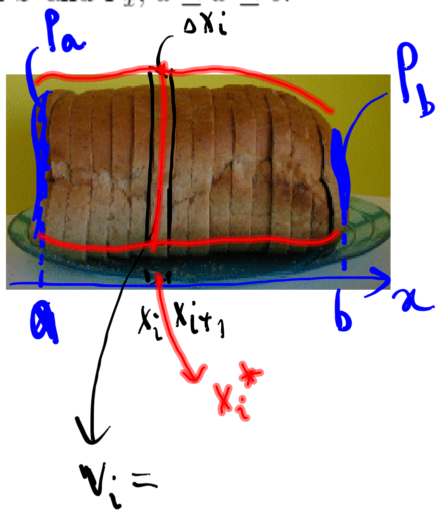


Let S be any solid. The intersection of S with a plane is a plane region that is called a **cross-section** of S.

P_x ia a plane perpendicular to x-axis and passing through x.

$A(x)$ is the area of cross-section obtained as intersection of S and P_x, $a \le x \le b$.





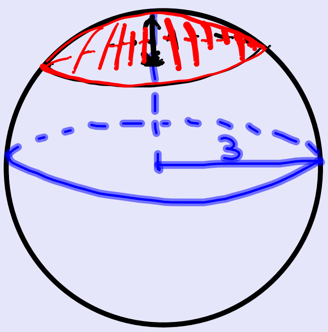
(Think of slicing a loaf of bread.)

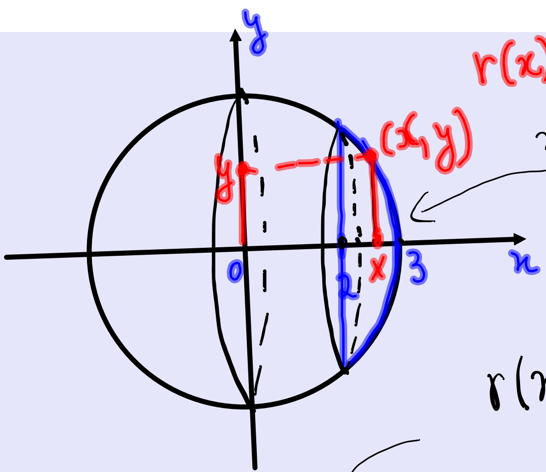
DEFINITION 1. *Let S be a solid that lies between the planes P_a and P_b. Then the volume of S is*

$$V = \underbrace{\lim_{\|P\|\to 0}} \sum_{i=1}^{n} A(x_i^*)\Delta x_i = \int_a^b A(x)\,dx$$

Important to remember: $A(x)$ is the area of a moving cross-section obtained by slicing through x **perpendicular** to the x-axis.

EXAMPLE 2. *Find the volume of the cap of a ball with radius 3 and height 1.*





$r(x) = y$

$x^2 + y^2 = 9 \Rightarrow$

$\Rightarrow y = \pm\sqrt{9-x^2}$

$r(x) = \sqrt{9-x^2}$

$$V = \int_2^3 A(x)\,dx = \int_2^3 \pi(\text{radius})^2\,dx = \int_2^3 \underbrace{\pi\left(\sqrt{9-x^2}\right)^2}_{A(x)}\,dx$$

$$= \pi\int_2^3 (9-x^2)\,dx = \pi\left(9x - \frac{x^3}{3}\right)\Big|_2^3 =$$

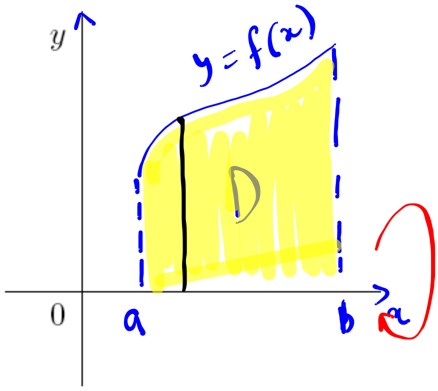
$$= \dots = \frac{8\pi}{3}$$

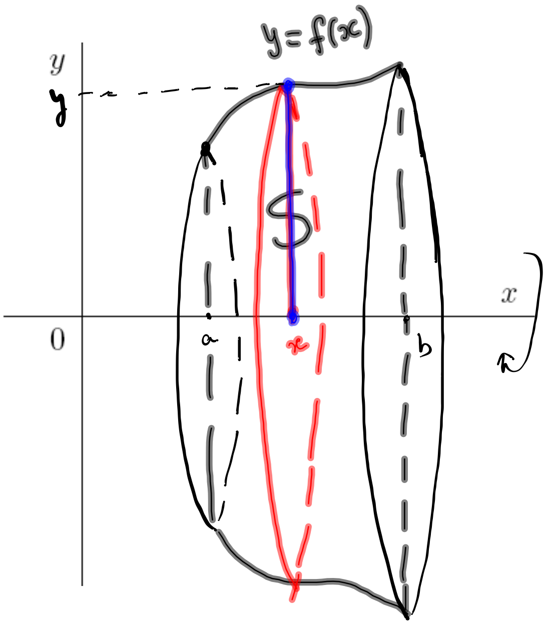
Volumes of Solids of Revolution (Disk Method)

Consider the plane region D bounded by the curves $y = f(x) \geq 0, y = 0, x = a, x = b$, i.e.

$$D = \{(x,y) : a \leq x \leq b,\ 0 \leq y \leq f(x)\}$$

Rotate D about a given axis to get the **solid of revolution** S:





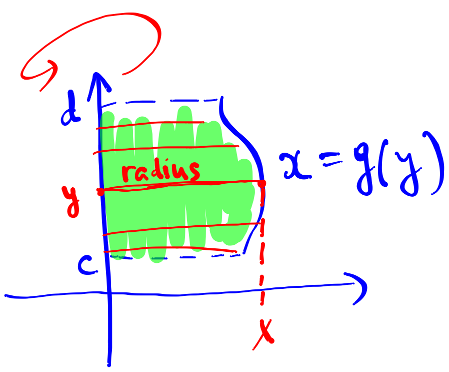
PROBLEM: `Determine the volume of solid of revolution.`

Solution: Using cross-sectional areas (**disk method**)

$$V = \int_a^b A(x)\,dx = \int_a^b \pi\,(\underbrace{\text{radius}}_{\text{distance between the curve } y=f(x) \text{ and the axis of rotation.}})^2 dx = \int_a^b \pi y^2 dx$$

$$\boxed{V = \pi \int_a^b [f(x)]^2 dx}$$

Rotation w.r.t. the y-axis



$$V = \int_c^d A(y)\,dy =$$

$$= \pi \int_c^d (\text{radius})^2 dy$$

$$\boxed{V = \pi \int_c^d [g(y)]^2 dy}$$

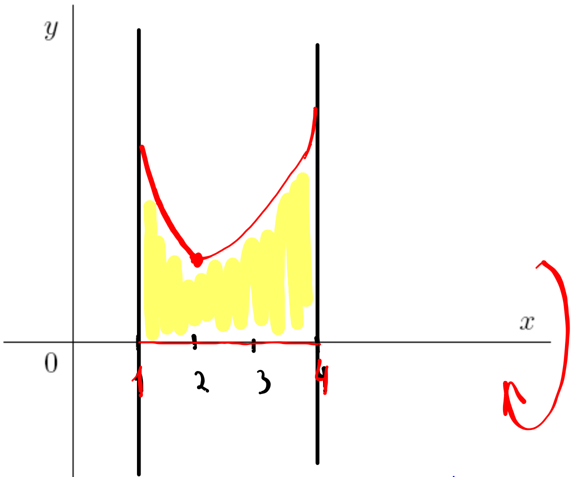
EXAMPLE 3. *Determine the volume of the solid obtained by rotating the region*

$$D = \{(x,y) : 1 \le x \le 4, 0 \le y \le x^2 - 4x + 5\}$$

about the x-axis.

$y = x^2 - 4x + 5 = f(x)$

$(2, 1)$ vertex



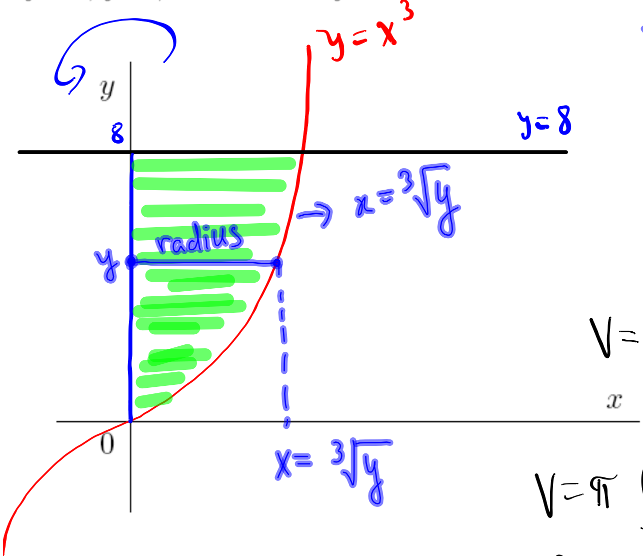
$$V = \int_1^4 A(x)\,dx = \pi \int_1^4 [f(x)]^2\,dx =$$

$$= \pi \int_1^4 (x^2 - 4x + 5)^2\,dx =$$

$$= \pi \int_1^4 x^4 + 16x^2 + 25 - 8x^3 + 10x^2 - 40x\,dx =$$

$$= \dots = \frac{78\pi}{5}$$

EXAMPLE 4. *Determine the volume of the solid obtained by rotating the region enclosed by $y = x^3$, $y = 8$, $x = 0$ about the y-axis.*



$$V = \int_0^8 A(y)\,dy$$

$$V = \pi \int_0^8 [g(y)]^2\,dy$$

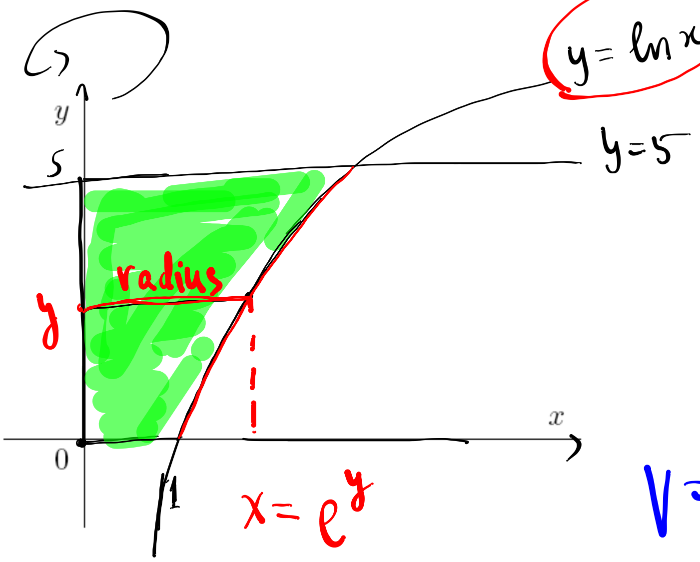
$$V = \pi \int_0^8 [\sqrt[3]{y}]^2\,dy$$

$$V = \pi \int_0^8 y^{\frac{2}{3}}\,dy$$

$$V = \pi \left. \frac{y^{\frac{2}{3}+1}}{\frac{2}{3}+1} \right|_0^8 = \dots = \frac{96\pi}{5}$$

EXAMPLE 5. *Determine the volume of the solid obtained by rotating the region enclosed by* $y = \ln x$, $y = 0$, $y = 5$ $x = 0$ *about the* y*-axis.*

$\Rightarrow x = e^y$



$$V = \int_0^5 \pi r^2 \, dy$$

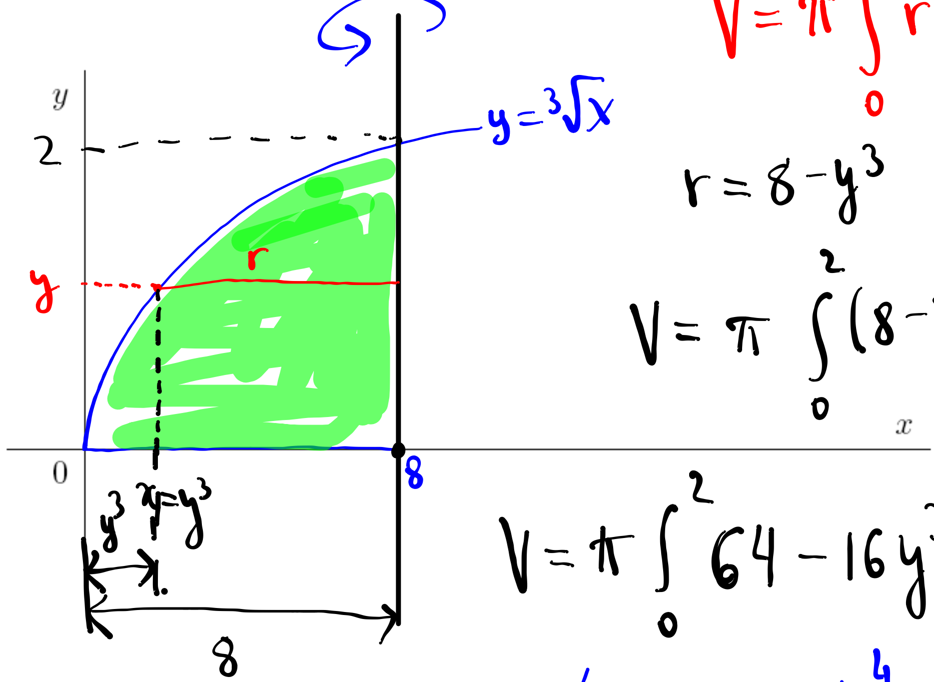
$$V = \pi \int_0^5 (e^y)^2 \, dy$$

$$V = \pi \int_0^5 e^{2y} \, dy$$

$$V = \pi \frac{1}{2} e^{2y} \Big|_0^5 = \frac{\pi}{2}(e^{10} - e^0)$$

$$V = \frac{\pi}{2}(e^{10} - 1)$$

EXAMPLE 6. *Determine the volume of the solid obtained by rotating the region enclosed by the curves* $y = \sqrt[3]{x}, x = 8, y = 0$ *about the line* $x = 8$.



$$V = \pi \int_0^2 r^2 \, dy$$

$$r = 8 - y^3$$

$$V = \pi \int_0^2 (8 - y^3)^2 \, dy$$

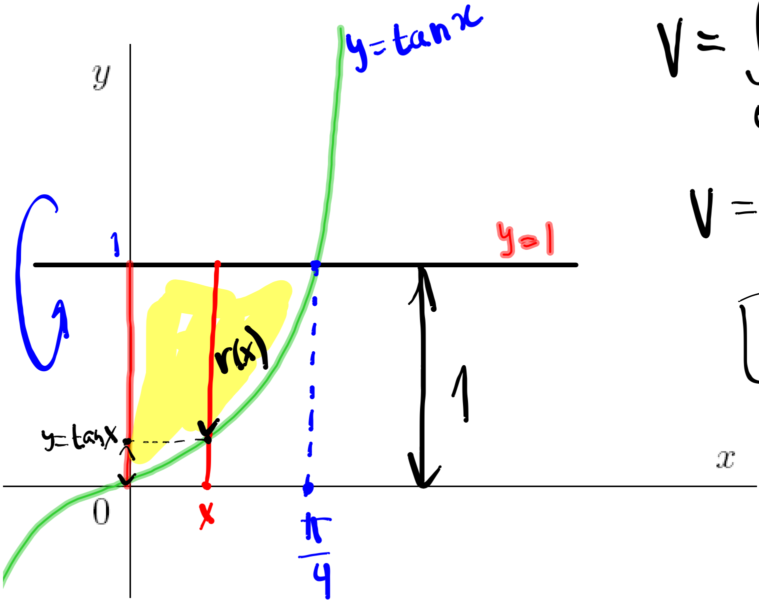
$$V = \pi \int_0^2 64 - 16y^3 + y^6 \, dy$$

$$V = \pi \left(64y - 16\frac{y^4}{4} + \frac{y^7}{7} \right)\Big|_0^2$$

...

$$V = \frac{576\pi}{7}$$

EXAMPLE 7. *Determine the volume of the solid obtained by rotating the region enclosed by* $y = \tan x$, $y = 1$ *and the* y*-axis about the line* $y = 1$.



$$V = \int_0^{\pi/4} A(x)\,dx$$

$$V = \pi \int_0^{\pi/4} [r(x)]^2\,dx$$

$$\boxed{r(x) = 1 - \tan x}$$

$$[r(x)]^2 = (1-\tan x)^2 = \underbrace{1 + \tan^2 x}_{\sec^2 x} - 2\tan x$$

$$V = \pi \int_0^{\pi/4} \sec^2 x - 2\tan x\,dx =$$

$$= \pi\left[\int_0^{\pi/4} \sec^2 x\,dx - 2\int_0^{\pi/4} \frac{\sin x}{\cos x}\,dx\right] =$$

use u-subs,
$u = \cos x$
$du = -\sin x\,dx$
$x = 0 \Rightarrow u = 1$
$x = \frac{\pi}{4} \Rightarrow u = \frac{\sqrt{2}}{2}$

$$= \pi\left[\tan x\Big|_0^{\pi/4} - 2\int_1^{\sqrt{2}/2} -\frac{du}{u}\right] = \pi(1 - \ln 2)$$

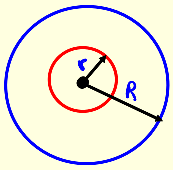
SUMMARY (Disk Method)

- Rotation about a horizontal axis $(y = k)$: $V = \displaystyle\int_a^b A(x)\,dx$
  $A(x) = \pi[r(x)]^2$
- Rotation about a vertical axis $(x = k)$: $V = \displaystyle\int_a^b A(y)\,dy$
  $A(y) = \pi[r(y)]^2$
- *Cross sections are orthogonal to the axis of rotating.* $\perp$

Washer Method

Use it when the cross-sections orthogonal to the axis of rotating of a solid of revolution are in the shape of a washer (ring).

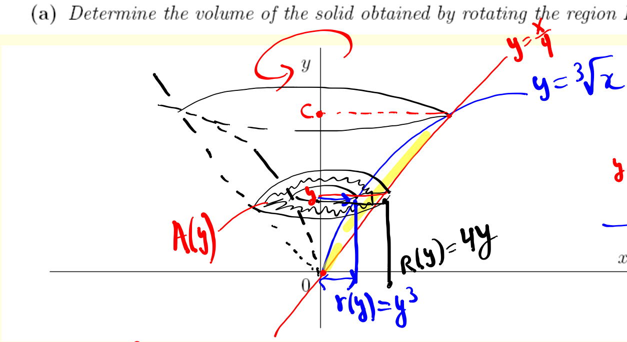
The area of a ring:

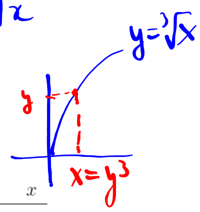


$$A = \pi R^2 - \pi r^2 = \pi (R^2 - r^2) =$$
$$= \pi \left(\begin{pmatrix} \text{outer} \\ \text{radius} \end{pmatrix}^2 - \begin{pmatrix} \text{inner} \\ \text{radius} \end{pmatrix}^2 \right)$$

EXAMPLE 8. *Let D be the plane region that lies in the first quadrant and enclosed by $y = \sqrt[3]{x}$ and $y = \frac{x}{4}$.*

(a) *Determine the volume of the solid obtained by rotating the region D about the y-axis.*





$$V = \int_0^c A(y)\,dy = \pi \int_0^c [R(y)]^2 - [r(y)]^2\,dy$$

inner radius
$r(y)$ = distance between the inner curve and the axis of rotation $= y^3$

outer radius
$R(y)$ = distance between the outer curve and the axis of rotation $= 4y$

Find c: $\begin{matrix} y=\sqrt[3]{x} \\ y = \frac{x}{4} \end{matrix} \rightarrow \begin{matrix} x = y^3 \\ x = 4y \end{matrix} \rightarrow y^3 = 4y$

$y(y^2-4)=0$

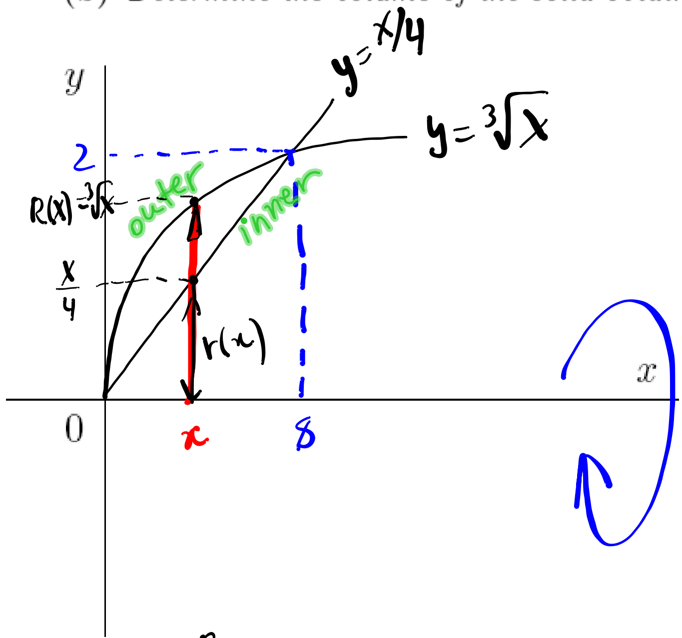
$y(y-2)(y+2)=0$

$y=0$ or $y=2$, or $y=-2$

c

$$V = \pi \int_0^2 (4y)^2 - (y^3)^2\,dy = \ldots = \frac{512\pi}{21}$$

(b) *Determine the volume of the solid obtained by rotating the region D about the x-axis.*



$$V = \int_0^8 A(x)\,dx$$

$$V = \pi \int_0^8 [R(x)]^2 - [r(x)]^2\,dx$$

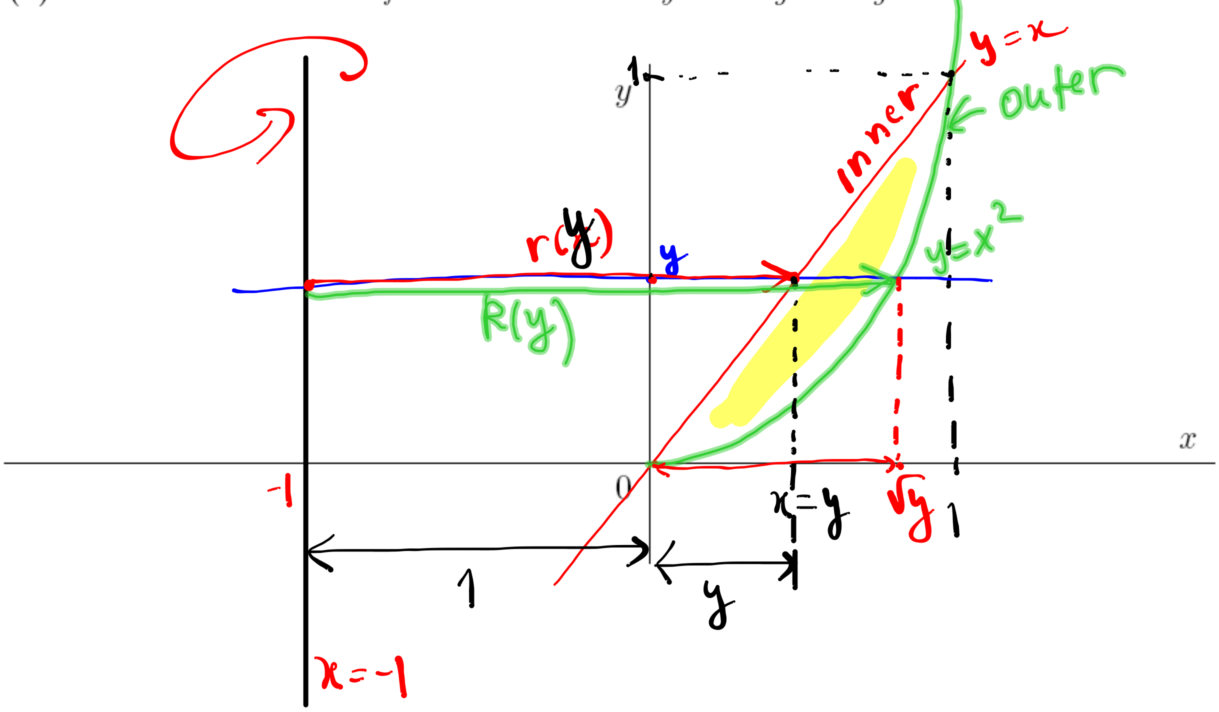
$$r(x) = \frac{x}{4}$$

$$R(x) = \sqrt[3]{x}$$

$$V = \pi \int_0^8 \left(\sqrt[3]{x}\right)^2 - \left(\frac{x}{4}\right)^2 dx = \pi \int_0^8 x^{\frac{2}{3}} - \frac{x^2}{16}\,dx = \ldots = \frac{128\pi}{15}$$

EXAMPLE 9. *Let D be the region enclosed by $y = x$ and $y = x^2$.*

(a) *Determine the volume of the solid obtained by rotating the region D about the line $x = -1$.*



$$V = \int_0^1 A(y)\,dy = \pi \int_0^1 [R(y)]^2 - [r(x)]^2\,dy$$

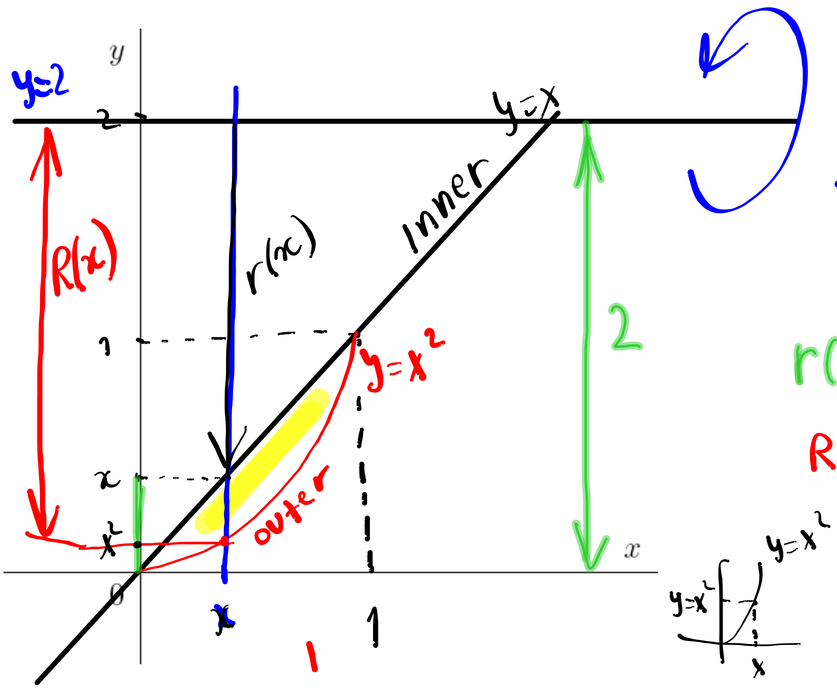
$r(y) = 1 + y$ $\qquad$ $R(y) = 1 + \sqrt{y}$

$$V = \pi \int_0^1 (1+\sqrt{y})^2 - (1+y)^2\,dy$$

$$V = \pi \int_0^1 (1 + y + 2\sqrt{y}) - (1 + y^2 + 2y)\,dy$$

$$V = \frac{\pi}{2}$$

(b) *Determine the volume of the solid obtained by rotating the region D about the line $y = 2$.*



$$V = \int_0^1 A(x)\,dx$$

$$V = \pi \int_0^1 [R(x)]^2 - [r(x)]^2\,dx$$

$$r(x) = 2 - x$$

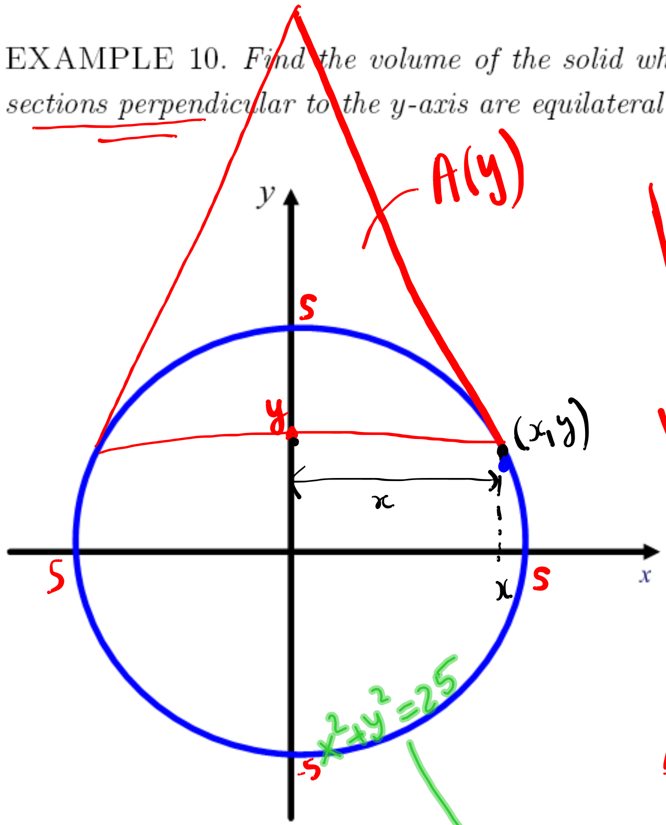
$$R(x) = 2 - x^2$$

$$V = \pi \int_0^1 (2-x^2)^2 - (2-x)^2\,dx = \ldots = \frac{8\pi}{15}$$

More general case: Cross Sections other than Circles

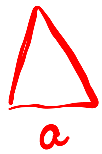
Use the basic formula: $$V = \int_a^b A(x)\,dx$$

EXAMPLE 10. *Find the volume of the solid whose base is a disk with radius 5 and the cross sections perpendicular to the y-axis are equilateral triangles.*



$$V = \int_{-5}^{5} A(y)\,dy,$$

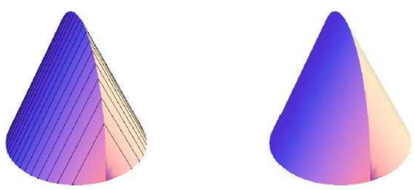
where $A(y)$ is area of equilateral triangle with base $a = 2x$



$$A(y) = \frac{a^2\sqrt{3}}{4} = \frac{(2x)^2\sqrt{3}}{4}$$

$x^2 = 25 - y^2$

$$A(y) = x^2\sqrt{3} = (25-y^2)\sqrt{3}$$

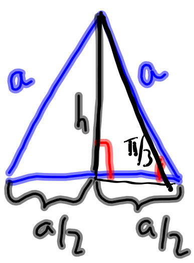


http://demonstrations.wolfram.com/SolidsOfKnownCrossSection/

Finally, $V = \int_{-5}^{5} (25-y^2)\sqrt{3}\,dy =$

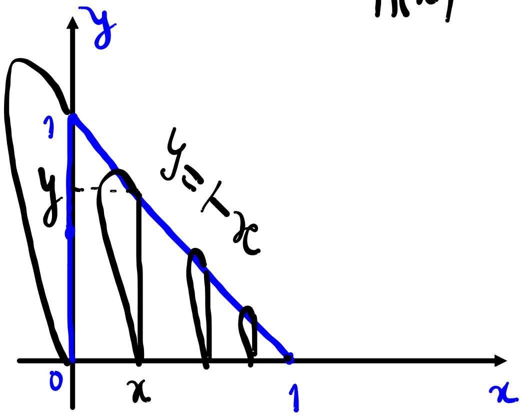
$$= 2\sqrt{3}\int_0^5 (25-y^2)\,dy = \ldots = \frac{500\sqrt{3}}{3}$$

Formula for area of equilateral Δ



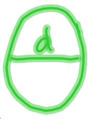
$$\left.\begin{array}{l} A = \dfrac{a\,h}{2} \\ h = a \sin\dfrac{\pi}{3} = \dfrac{a\sqrt{3}}{2} \end{array}\right\} \Rightarrow A = \frac{a \cdot \frac{a\sqrt{3}}{2}}{2} \Rightarrow \boxed{A = \frac{a^2\sqrt{3}}{4}}$$

EXAMPLE 11. *The base of the solid S is the triangular region with the vertices $(0,0),(1,0)$ and $(0,1)$. Find the volume of S if the cross sections perpendicular to the x-axis are semicircles with diameters on the base.* $= A(x)$



$$V=\int_0^1 A(x)\,dx$$

$$A=\frac{\pi d^2}{4}$$



$A(x)$ is area of semicircle with diameter $= y = 1-x$

radius $= \dfrac{1-x}{2} = \dfrac{\text{diameter}}{2}$

$$A(x)=\frac{1}{2}\pi(\text{radius})^2=\frac{\pi}{2}\left(\frac{1-x}{2}\right)^2$$

$$A(x)=\frac{\pi}{8}(1-x)^2$$

$$V=\int_0^1 \frac{\pi}{8}(1-x)^2\,dx=\frac{\pi}{8}\int_0^1 1+x^2-2x\,dx=$$

$$=\frac{\pi}{8}\left(x+\frac{x^3}{3}-x^2\right)\Big|_0^1=$$

$$=\frac{\pi}{8}\left(1+\frac{1}{3}-1\right)=\frac{\pi}{24}$$