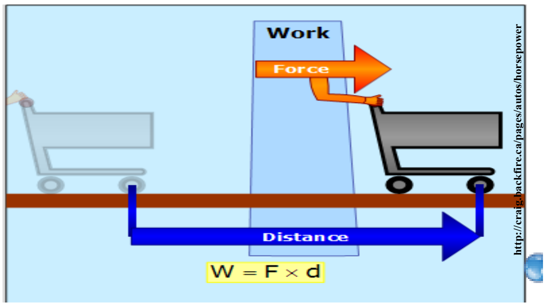
7.4: Work

PROBLEM: Find the amount of work that is done by a force in moving an object.

- Case 1: constant force.



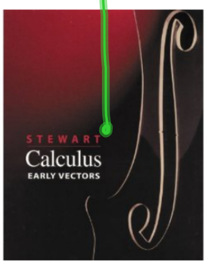
Work W done in moving an object a distance d meters is given by

$$W = Fd.$$

In the SI metric system: $[J] = [N][m]$

In the British engineering system: [ft][lb]. Also $1 ft\text{-}lb \approx 1.36J$.

EXAMPLE 1. *How much work is done in lifting your Calculus book* ($\overset{m}{\underline{2.1kg}}$) *off the floor to put it on a desk that is* $0.6m$ *high.*

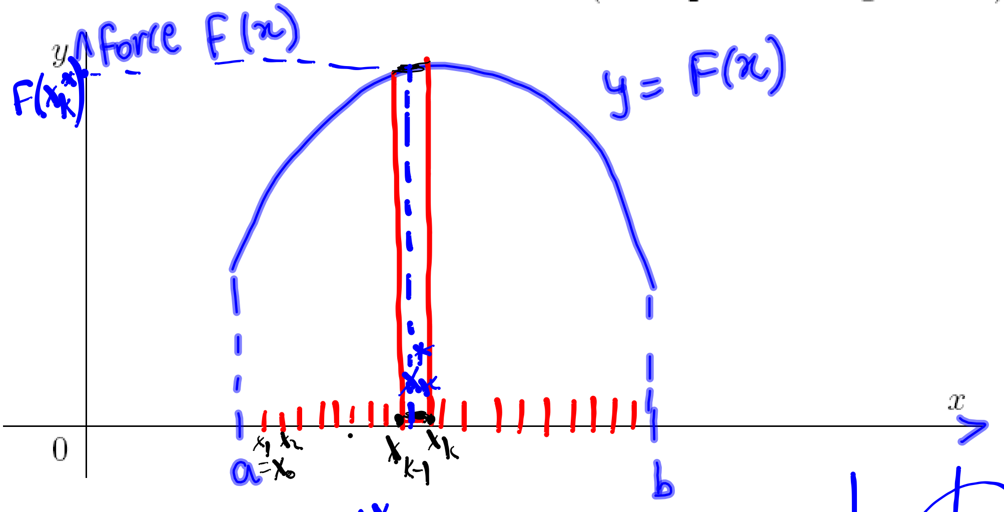


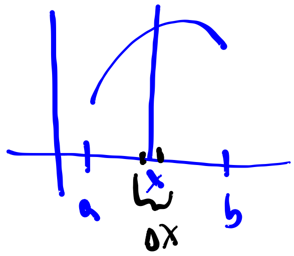
d

$$W = Fd = mg \cdot d$$

$$W = 2.1 \cdot 9.8 \cdot 0.6 = 12.348 \text{ J}$$

- Case 2: <u>non constant force.</u> (It requires integration.)





$$W_k = F(x_k^*)\overbrace{(x_k - x_{k-1})}^{\Delta x_k} = F(x_k^*)\Delta x_k$$

$$W \approx \sum_{k=1}^{n} W_k = \sum_{k=1}^{n} F(x_k^*)\Delta x_k$$

$W_x = F(x)\,\Delta x$

$W \approx \sum_{a \le x \le b} W_x \Rightarrow \int_a^b F(x)dx$

Finally, $W = \lim_{\|P\|\to 0} \sum_{k=1}^{n} F(x_k^*)\Delta x_k$ where $\|P\| = \max_k \Delta x_k$.

Thus, *work done in moving an object from $x = a$ to $x = b$ is*

$$\boxed{W = \int_a^b F(x)\,\mathrm{d}x}$$

EXAMPLE 2. *When a particle is at distance x feet from the origin, a force of $3x^2 + 2x$ pounds acts on it. How much work is done in moving it from $\underbrace{x = 1}_{a}$ to $\underbrace{x = 3}_{b}$ along the x-axis?*

$F(x) = 3x^2 + 2x$

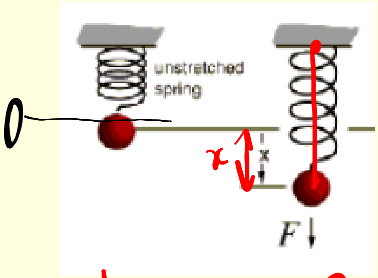
$$W = \int_1^3 F(x)dx = \int_1^3 3x^2 + 2x\, dx = x^3 + x^2 \Big|_1^3$$

$W = 34$ ft-lb

EXAMPLE 3. *A spring has a natural length of* $1m$. *If a* $50N$ *force is required to keep it stretched to a length* $3m$, *how much work is done in stretching the spring from* $2m$ *to* $5m$?

Solution By Hooke's law the force required to stretch a spring x units **beyond** its natural length is

$$F = kx$$



Reformulate the problem in Example 3:
If a 50N force is required to keep a spring stretched 2 m beyond its natural length, how much work is done in stretching the spring from 1 m to 4 m beyond its natural length?

Solution of Example 3.

Find k:

$F=50$, $x=2$ } $F = kx$

$50 = k\cdot 2$

$\boxed{k=25}$

$F(x) = 25x$

$$W = \int_1^4 F(x)\,dx = \int_1^4 25x\,dx = 25\frac{x^2}{2}\Big|_1^4 = 187.5\ J$$

EXAMPLE 4. *If the work required to stretch a spring 1ft beyond its natural length is 12ft-lb, how much work is needed to stretch it 9 inches beyond its natural length?*

$$9\text{ in} = \frac{9}{12}\text{ ft} = \frac{3}{4}\text{ ft}$$

$$W = \int_a^b kx\,dx$$

$$12 = \int_0^1 kx\,dx$$

$$12 = k\int_0^1 x\,dx = \frac{1}{2}$$

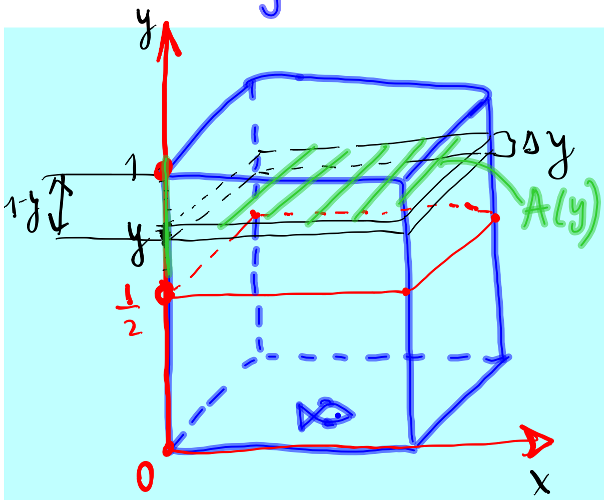
$$12 = \frac{k}{2} \Rightarrow \boxed{k=24}$$

Find $W = \int_0^{3/4} kx\,dx$

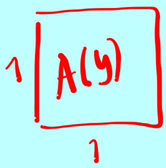
$$W = \int_0^{3/4} 24x\,dx$$

$$W = 24\frac{x^2}{2}\Big|_0^{\frac{3}{4}} = \boxed{\frac{27}{4}\text{ ft-lb}}$$

EXAMPLE 5. *An aquarium has a form of a cube whose side is* 1m. *If the aquarium is full of water, find the work needed to pump* 50% *of the water out of the aquarium. (The density of water is* $1000kg/m^3$.) $= \rho$



Work to pump out the layer of water at y with thickness Δy



$$W_y = (\text{Force})\cdot(\text{distance})$$

$$W_y = mg\cdot(1-y)$$

$$W_y = \rho V(y)\, g\,(1-y)$$

$$W_y = \rho\, \underbrace{A(y)}_{1}\cdot \Delta y \cdot g(1-y)$$

$$W_y = \rho g (1-y)\Delta y$$

$$W \approx \sum_{\frac{1}{2}\le y\le 1} W_y = \sum_{\frac{1}{2}\le y\le 1} \rho g (1-y)\Delta y$$

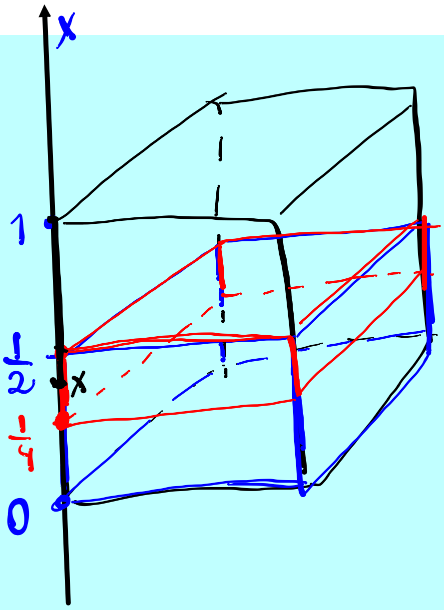
$A(x)=1^2=1$

$$W = \int_{\frac{1}{2}}^{1} \rho g (1-y)\,dy = \rho g \int_{\frac{1}{2}}^{1} (1-y)\,dy =$$

$$= 1000\cdot 9.8 \left(y-\frac{y^2}{2}\right)\Big|_{\frac{1}{2}}^{1} = \boxed{1225\,J}$$

Remark: $W = \rho g \int_{\text{lower slice level}}^{\text{top slice level}} \left(\text{cross sectional area of slice}\right)\left(\text{depth of slice from the top}\right) d\,\sqcup$

EXAMPLE 6. *Work the previous example assuming the aquarium is only 1/2 full.*

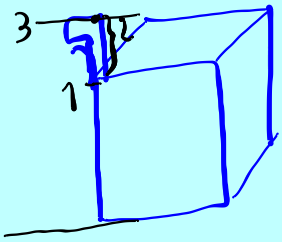


$$W = \rho g \int_{1/4}^{1/2} \underbrace{A(x)}_{=1} \cdot (1-x)\, dx$$

$$W = \rho g \int_{1/4}^{1/2} (1-x)\, dx$$

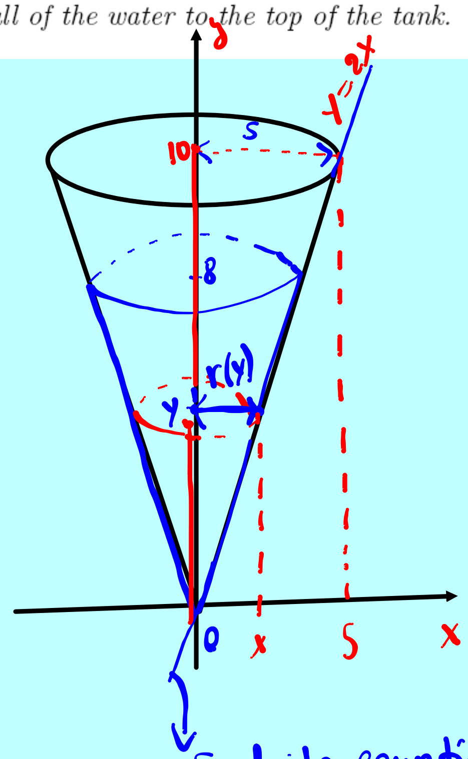
$$W = 1000 \cdot 9.8 \left(x - \frac{x^2}{2} \right) \Bigg|_{1/4}^{1/2} = \boxed{1531.25}$$

Remark



$$W = \rho g \int A(x) \cdot (1+2-x)\, dx$$

EXAMPLE 7. *A tank has a shape of an inverted circular cone with height* $10m$ *and base radius* $5m$. *It is filled with water to a height of* $8m$. *Find the work required to empty the tank by pumping all of the water to the top of the tank. (The density of water is* $1000kg/m^3$.*)*



Find its equation

$$W = \rho g \int_0^8 A(y)\,(10-y)\,dy$$

$$\left.\begin{aligned} A(y) &= \pi [r(y)]^2 \\ r(y) &= x = \frac{y}{2} \end{aligned}\right\} \Rightarrow A(y) = \pi\left(\frac{y}{2}\right)^2, \quad A(y) = \frac{\pi y^2}{4}$$

$$\left.\begin{aligned} y &= mx \\ m &= \frac{10}{5} = 2 \end{aligned}\right\} \Rightarrow y = 2x$$

$$W = \rho g \int_0^8 \frac{\pi y^2}{4}(10-y)\,dy$$

$$W = \frac{\rho g \pi}{4}\int_0^8 10y^2 - y^3\,dy$$

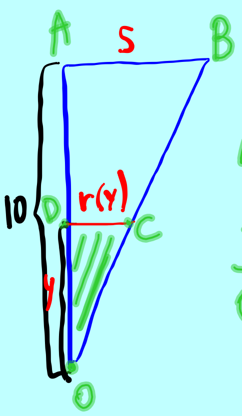
$$W = \frac{1000 \cdot 9.8\,\pi}{4}\left(\frac{10y^3}{3} - \frac{y^4}{4}\right)\Bigg|_0^8$$

$$\boxed{W = 5.3 \cdot 10^6\ J}$$

Final answer

Remark Another Way to get the radius $r(y)$.

Use similar triangles



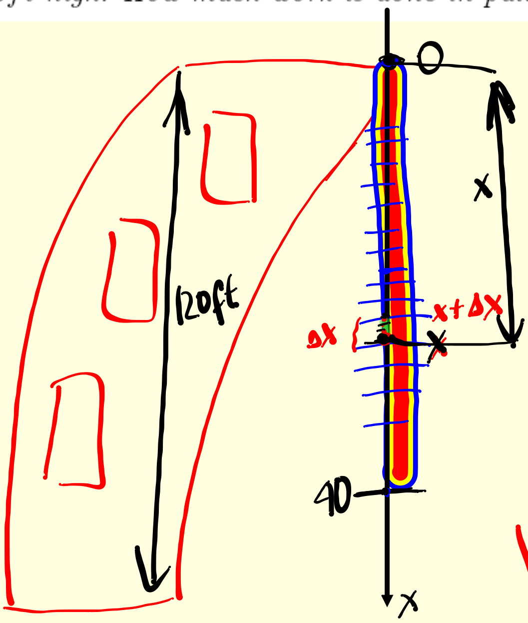
$$\Delta ODC \sim \Delta OAB$$

$$\frac{OA}{OD} = \frac{AB}{DC} \Rightarrow \frac{10}{y} = \frac{5}{r(y)}$$

$$10\,r(y) = 5y$$

$$r(y) = \frac{5}{10}y = \frac{y}{2}.$$

EXAMPLE 8. *A heavy rope 40ft long, weighs 0.4lb/ft and hangs over the edge of a building 120ft high. How much work is done in pulling the rope to the top of the building?*



$$W = (\text{force})(\text{distance})$$

force = weight

$$W_x = \begin{pmatrix}\text{weight}\\ \text{of slice}\\ \text{length } \Delta x\end{pmatrix} \cdot \begin{pmatrix}\text{lifting}\\ \text{distance}\\ \text{from the top}\end{pmatrix}$$

$$W_x = (0.4\,\Delta x)(x)$$

Total work $$W \approx \sum_{0 \le x \le 40} W_x = \sum_{0 \le x \le 40} 0.4x\,\Delta x$$

$$W = \int_0^{40} 0.4\,x\,dx = 0.4\,\frac{x^2}{2}\Big|_0^{40} = \boxed{320\text{ ft-lb}}$$

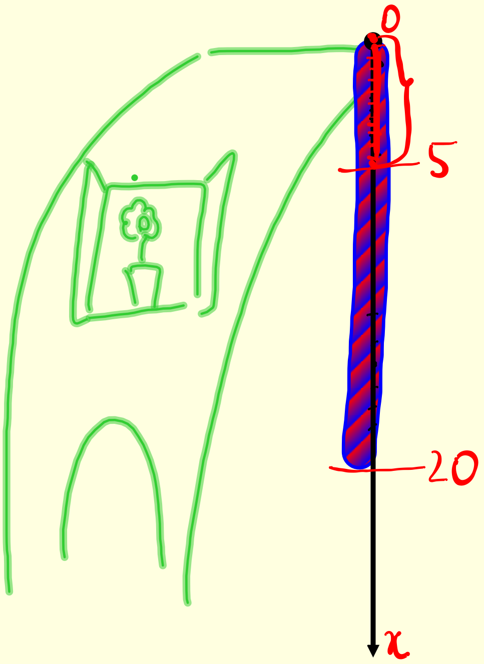
REMARK 10. The exact height of the building doesn't matter.

Remark:

$$W = \int_{\text{min distance from the top}}^{\text{max distance from the top}} \begin{pmatrix}\text{slice}\\ \text{weight}\\ \Delta x\end{pmatrix} \cdot \begin{pmatrix}\text{lifting}\\ \text{distance}\\ \text{from top}\end{pmatrix} dx$$

Note: this formula is w.r.t. origin as above

EXAMPLE 9. *A uniform cable hanging over the edge of a tall building is* $20ft$ *long and weight* $30lb$. *How much work is required to pull* $5ft$ *of the cable to the top?*



weight $= \frac{30}{20}\, lb/ft = \frac{3}{2}\, lb/ft$

$$W = W_1 + W_2$$

W_1: work to pull top 5ft of the cable

need integration

W_2: work to pull bottom 15ft of the cable

$$W_2 = \begin{pmatrix}\text{weight}\\ \text{of 15ft}\end{pmatrix} \cdot \begin{pmatrix}\text{distance}\\ = 5ft\end{pmatrix}$$

$$W_2 = \left(15 \cdot \frac{3}{2}\right) \cdot 5 = \frac{225}{2}\, ft\text{-}lb$$

$$W_1 = \int_0^5 \frac{3}{2} x \, dx = \frac{3}{2} \frac{x^2}{2}\Big|_0^5 = \frac{3}{4} \cdot 25 = \frac{75}{4}\, ft\text{-}lb$$

$$W = \frac{75}{4} + \frac{225}{2} = \frac{525}{4}\, ft\text{-}lb$$

REMARK 10. The exact height of the building doesn't matter.