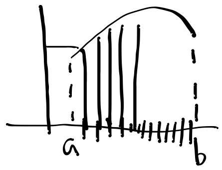
7.5: Average Value of a Function

The average value of finitely many numbers $y_1, y_2, \ldots, y_n$:

$$\bar{y} = \frac{y_1 + y_2 + \ldots + y_n}{n}$$

The average value of a function $y = f(x)$ over the interval $[a, b]$:



$$\boxed{f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx.}$$

EXAMPLE 1. *Determine the average value of* $f(x) = x^2 - 4x + 7\sin(\pi x)$ *over the interval* $[-\frac{1}{2}; \frac{1}{2}]$.

$$f_{ave} = \underbrace{\frac{1}{\frac{1}{2} - (-\frac{1}{2})}}_{=1} \int_{-\frac{1}{2}}^{\frac{1}{2}} (x^2 - 4x + 7\sin \pi x)\,dx$$

$$f_{ave} = \int_{-\frac{1}{2}}^{\frac{1}{2}} x^2\,dx - 4\underbrace{\int_{-\frac{1}{2}}^{1/2} x\,dx}_{=0} + 7\underbrace{\int_{-\frac{1}{2}}^{1/2} \sin \pi x\,dx}_{=0}$$

$$f_{ave} = 2\int_0^{\frac{1}{2}} x^2\,dx = 2\,\frac{x^3}{3}\Big|_0^{\frac{1}{2}} = \frac{2}{3}\cdot\frac{1}{8} = \boxed{\frac{1}{12}}$$

Remark symmetric

If f is $[-a, a]$ and

1) $f(-x) = -f(x)$ odd

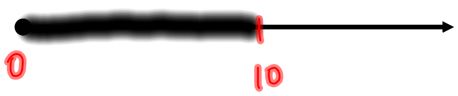
then $\int_{-a}^{a} f(x)\,dx = 0$

2) $f(-x) = f(x)$ even

$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

EXAMPLE 2. *The temperature of a metal rod, 10 m long, is* $5x$ *(in °C) at a distance* x *meters from one end of the rod. What is the average temperature of the rod?*

$T(x) = 5x$, $0 \le x \le 10$



$$T_{ave} = \frac{1}{10-0}\int_0^{10} 5x\,dx$$

$$T_{ave} = \frac{1}{10}\cdot 5\,\frac{x^2}{2}\Big|^{10} = \frac{1}{4}\cdot 10^2 = 25\,^\circ C$$

EXAMPLE 3. *Find the value(s) b s. t. the average value of $f(x) = 3 + 2x - 3x^2$ on the interval $[0, b]$ is equal to* 1.

$\Downarrow$

$b > 0$

$$f_{ave} = 1 \quad \text{on} \quad [0, b]$$

$$\frac{1}{b-0} \int_0^b (3 + 2x - 3x^2)\, dx = 1$$

$$\frac{1}{b} \left(3x + x^2 - x^3 \right) \Big|_0^b = 1$$

$$\frac{1}{b} \left(3b + b^2 - b^3 \right) = 1$$

$$3 + b - b^2 = 1, \quad b \neq 0$$

$$b^2 - b - 2 = 0$$

$$(b+1)(b-2) = 0$$

Final answer

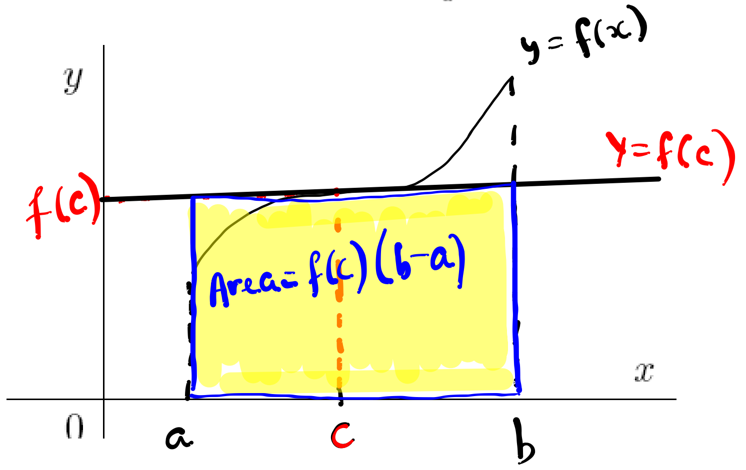
$$b = -1 < 0 \quad \text{or} \quad b = 2 > 0$$

MEAN VALUE THEOREM FOR INTEGRALS: *If f is continuous on $[a,b]$, then there exists a number c on $[a,b]$ s.t.*

$$\int_a^b f(x)\,\mathrm{d}x = f(c)(b-a). = \text{Area}$$

$$\Downarrow$$

$$f(c) = \underbrace{\frac{1}{b-a}\int_a^b f(x)\,dx}_{f_{ave}}$$



READ IT!

The geometric interpretation of the Mean Value Theorem for Integrals: for *positive* functions f, there is a number c s.t. the rectangle with base $[a,b]$ and height $f(c)$ has the same area as the region under the graph of f from a to b.

EXAMPLE 4. *If g is continuous and $\int_{-1}^{7} g(x)\,\mathrm{d}x = 24$ show that g takes on the value 3 at least once on the interval $[-1, 7]$.*

Show that there exist c such that $g(c) = 3$.

Use Mean Value Theorem:

g is cont. on $[-1, 7]$ $\Rightarrow$ there exists c

such that

$$\underbrace{\int_{-1}^{7} g(x)\,dx}_{=24} = g(c)\big(\overbrace{7-(-1)}^{8}\big)$$

$$24 = 8g(c) \Rightarrow g(c) = \frac{24}{8} = \boxed{3}$$

EXAMPLE 5. *Determine the number c that satisfies the Mean Value Theorem for Integrals for the function $f(x) = x^2 - 2x - 2$ on the interval [1,4]*

$$\int_1^4 f(x)\,dx = f(c)(4-1), \qquad 1<c<4$$

$$\int_1^4 (x^2-2x-2)\,dx = 3(c^2-2c-2)$$

$$\underbrace{\left.\frac{x^3}{3} - x^2 - 2x\right|_1^4}_{\begin{array}{c}\| \\ -\frac{8}{3}-\left(-\frac{8}{3}\right)=0\end{array}} = 3(c^2-2c-2)$$

CHECK IT!

$$3(c^2-2c-2)=0$$

$$c^2-2c-2=0$$

$$ax^2+bx+c=0$$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$c_{1,2} = \frac{2 \pm \sqrt{4+8}}{2} = \frac{2 \pm 2\sqrt{3}}{2}$$

$$c_1 = 1+\sqrt{3}$$

$$c_2 = 1-\sqrt{3}$$

$1 < 1+\sqrt{3} < 4$, but $1-\sqrt{3} < 1$

Final answer $\boxed{c = 1+\sqrt{3}}$