8.1: Integration by Parts

- $\int x^n e^{kx}\,\mathrm{d}x$
- $\int x^n \sin(kx)\,\mathrm{d}x$, $\int x^n \cos(kx)\,\mathrm{d}x$
- $\int x^n \ln x\,\mathrm{d}x$
- $\int e^x \cos x\,\mathrm{d}x$, $\int e^x \sin x\,\mathrm{d}x$
- $\int \sec^n x\,\mathrm{d}x$ and some integrals involving inverse trigonometric functions.

The **integration by parts formula**:

$$\underbrace{\int fg'\,\mathrm{d}x} = fg - \int f'g\,\mathrm{d}x$$

Proof:

$$(fg)' = fg' + f'g$$

$$fg = \int (fg)'\,dx = \underbrace{\int fg'\,dx} + \int f'g\,dx$$

$$\int fg'\,dx = fg - \int f'g\,dx$$

Rewrite the above formula using the following substitutions:

$$u = f(x), \qquad v = g(x)$$

$$du = f'(x)\,dx \qquad dv = g'(x)\,dx$$

$$\int \underbrace{f}_{u}\,\underbrace{g'\,dx}_{dV} = \underbrace{f\,g}_{uv} - \int \underbrace{g}_{v}\,\underbrace{f'\,dx}_{du}$$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

LIATE rule to choose u and dETAIL rule to choose dv

Whichever function comes first in the list it should be u.

- **L** Logarithmic functions ln x, $\log_a x$,...
- **I** Inverse Trigonometric functions arctan x, arcsin x,...
- **A** Algebraic functions x, x^2, $66x^{15}$,...
- **T** Trigonometric functions sin x,cos x,sec x, ...
- **E** Exponential functions e^x, e^{-3x}, 2^x,...

EXAMPLE 1. *Evaluate* $I = \int x\cos(5x)\,dx$

$u = x \qquad dv = \cos 5x\,dx$

$du = dx \qquad v = \int dv = \int \cos 5x\,dx = \frac{1}{5}\sin 5x$

$I = uv - \int v\,du = x\frac{1}{5}\sin 5x - \int \frac{1}{5}\sin 5x\,dx$

$I = \frac{1}{5}x\sin 5x - \frac{1}{5}\left(-\frac{1}{5}\cos 5x\right) + C$

$I = \frac{x}{5}\sin 5x + \frac{1}{25}\cos 5x + C$

Note

if $F'(x) = f(x)$

$\Downarrow$

$\int f(ax)\,dx = \frac{1}{a}F(ax) + C$

EXAMPLE 2. *Evaluate* $I = \int xe^{kx}\,dx$

$u = x \qquad dv = e^{kx}\,dx$

$du = dx \qquad v = \frac{1}{k}e^{kx}$

$I = x\cdot\frac{1}{k}e^{kx} - \int \frac{1}{k}e^{kx}\,dx = \frac{x}{k}e^{kx} - \underbrace{\frac{1}{k^2}}_{\frac{1}{k}\cdot\frac{1}{k}}e^{kx} + C$

Integration by parts formula for definite integrals:

$$\int_a^b u\,\mathrm{d}v = uv|_a^b - \int_a^b v\,\mathrm{d}u,$$

where

$$uv|_a^b = u(b)v(b) - u(a)v(a).$$

EXAMPLE 3. *Evaluate* $I = \int_{-1}^{3} xe^{3x}\,\mathrm{d}x$

$u = x \qquad dv = e^{3x}dx$

$du = dx \qquad v = \frac{1}{3}e^{3x}$

$$I = uv\Big|_{-1}^{3} - \int_{-1}^{3} v\,du = \frac{1}{3}xe^{3x}\Big|_{-1}^{3} - \frac{1}{3}\int_{-1}^{3} e^{3x}\,dx$$

$$I = \frac{1}{3}\left[3e^{9} - (-1)e^{-3}\right] - \frac{1}{9}e^{3x}\Big|_{-1}^{3}$$

$$I = e^{9} + \frac{1}{3}e^{-3} - \frac{1}{9}e^{9} + \frac{1}{9}e^{-3}$$

$$I = \frac{8}{9}e^{9} + \frac{4}{9}e^{-3}$$

EXAMPLE 4. *Evaluate* $I = \int x^2 \sin(5x)\, dx$

1st

$$u = x^2 \qquad dv = \sin 5x\, dx$$

$$du = 2x\, dx \qquad v = \int dv = -\frac{1}{5}\cos 5x$$

$$I = uv - \int v\, du = -\frac{x^2}{5}\cos 5x + \frac{2}{5}\underline{\int x \cos 5x\, dx}$$

2nd

$$u = x \qquad dv = \cos 5x\, dx$$

$$du = dx \qquad v = \frac{1}{5}\sin 5x$$

$$I = -\frac{x^2}{5}\cos 5x + \frac{2}{5}\left[\frac{x}{5}\sin 5x - \frac{1}{5}\overbrace{\int \sin 5x\, dx}^{-\frac{1}{5}\cos 5x}\right]$$

$$I = -\frac{x^2}{5}\cos 5x + \frac{2}{25}x \sin 5x + \frac{2}{125}\cos 5x + C$$

Remark: We also can use so called TABULAR INTEGRATION BY PARTS:

sign	Differentiate	Integrate (find antiderivative)
+	$u = x^2$	$dv = \sin 5x\, dx$
$-$	$2x$	$-\frac{1}{5}\cos 5x$
+	2	$-\frac{1}{25}\sin 5x$
$-$	0	$\frac{1}{125}\cos 5x$

$+C$

$$I = x^2\left(-\frac{1}{5}\cos 5x\right) - 2x\left(-\frac{1}{25}\sin 5x\right) + 2\cdot\frac{1}{125}\cos 5x + C$$

EXAMPLE 5. *Evaluate* $I = \int \ln x \, dx$

$u = \ln x \qquad dv = dx$

$du = \frac{dx}{x} \qquad v = x$

$I = uv - \int v \, du = x \ln x - \int \cancel{x} \frac{dx}{\cancel{x}} = x \ln x - x + C$

EXAMPLE 6. *Evaluate* $I = \int x^9 \ln x \, dx$

$$u = \ln x \qquad dv = x^9\, dx$$

$$du = \frac{dx}{x} \qquad v = \frac{x^{10}}{10}$$

$$I = uv - \int v\, du = \frac{x^{10}}{10} \ln x - \int \frac{x^{10}}{10} \frac{dx}{x}$$

$$I = \frac{x^{10}}{10} \ln x - \frac{1}{10} \underbrace{\int x^9\, dx}_{\frac{x^{10}}{10}} = \frac{x^{10}}{10} \ln x - \frac{1}{100} x^{10} + C$$

EXAMPLE 7. *Evaluate* $I = \int \arcsin x \, dx$

Review derivative formulas for inverse trig. func.

$u = \arcsin x \qquad dv = dx$

$du = \dfrac{dx}{\sqrt{1-x^2}} \qquad v = x$

$I = x \arcsin x - \underbrace{\int x \dfrac{dx}{\sqrt{1-x^2}}}$

Use u-subst.

$u = 1 - x^2$

$du = -2x\,dx$

$x\,dx = -\frac{1}{2} du$

$I = x \arcsin x + \frac{1}{2} \underbrace{\int \dfrac{du}{\sqrt{u}}} \qquad \int u^{-\frac{1}{2}} du$

$I = x \arcsin x + \frac{1}{\cancel{2}} \dfrac{u^{\frac{1}{2}}}{\frac{1}{\cancel{2}}} + C$

$I = x \arcsin x + \sqrt{1-x^2} + C$

EXAMPLE 8. *Evaluate* $I = \int_0^1 \arctan x \, dx$

$u = \arctan x$ $\qquad dv = dx$

$du = \dfrac{dx}{1+x^2}$ $\qquad v = x$

$I = uv\Big|_0^1 - \int_0^1 v\,du = x \arctan x\Big|_0^1 - \boxed{\int_0^1 \dfrac{x\,dx}{1+x^2}}$

u-sub
$u = 1+x^2$
$du = 2x\,dx$
$x=0 \Rightarrow u=1$
$x=1 \Rightarrow u=2$

$I = 1 \cdot \arctan 1 - 0 \cdot \arctan 0 - \dfrac{1}{2}\int_1^2 \dfrac{du}{u}$

$I = \dfrac{\pi}{4} - \dfrac{1}{2}\ln u\Big|_1^2 = \dfrac{\pi}{4} - \dfrac{1}{2}\left[\ln 2 - \underbrace{\ln 1}_{=0}\right]$

$I = \dfrac{\pi}{4} - \dfrac{1}{2}\ln 2$

EXAMPLE 9. *Evaluate* $I = \int e^x \cos x \, dx$ Loop

1^{st} $u = \cos x$ $dv = e^x dx$

$du = -\sin x \, dx$ $v = e^x$

$$I = e^x \cos x + \underline{\int e^x \sin x \, dx}$$

2^{nd} $u = \sin x$ $dv = e^x dx$

$du = \cos x \, dx$ $v = e^x$

$$I = e^x \cos x + e^x \sin x - \int e^x \cos x \, dx \quad = I$$

$$2I = e^x \cos x + e^x \sin x$$

$$\boxed{I = \frac{1}{2} e^x (\cos x + \sin x) + C}$$

EXAMPLE 10. *A particle that moves along a straight line has velocity* $v(t) = t^2e^{-t}$ *meters per second after* t *seconds. How far will it travel during the first 10 seconds?* $s(10)$ - ?

$$s'(t) = v(t)$$

$$s(t) = \int_0^t v(\tau)\,d\tau$$

$$s(10) = \int_0^{10} \tau^2 e^{-\tau}\,d\tau$$

+	$u = \tau^2$	$dv = e^{-\tau}d\tau$
−	2τ	$-e^{-\tau}$
+	2	$e^{-\tau}$
−	0	$-e^{-\tau}$

$$s(10) = -\tau^2e^{-\tau} - 2\tau e^{-\tau} + (-2e^{-\tau})\Big|_0^{10}$$

$$s(10) = e^{-\tau}(-\tau^2 - 2\tau - 2)\Big|_0^{10}$$

$$s(10) = e^{-10}(-100 - 20 - 2) + 2$$

$$\boxed{s(10) = 2 - 122e^{-10}\ \text{m}}$$

Remark

$e^{-10} \approx 5 \cdot 10^{-5}$ (very small) $\Rightarrow s(10) > 0$

Also, t^2e^{-t} is positive $\Rightarrow \int t^2e^{-t} > 0$