

8.2: Trigonometric Integrals

Recall:

half-angle identities:

$$\tan^2 x = \sec^2 x - 1$$

$$\sin^2 x + \cos^2 x = 1$$

$$\tan^2 x + 1 = \sec^2 x$$

$$\cot^2 x + 1 = \csc^2 x$$

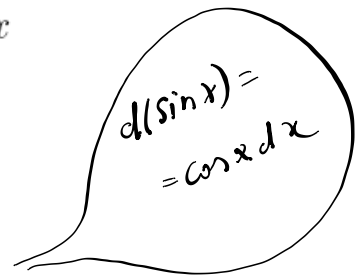
$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\sin x \cos x = \frac{1}{2} \sin 2x$$

EXAMPLE 1. Evaluate $I = \int \cos x \sin^{2011} x \, dx$

$$I = \int \underbrace{(\sin x)^{2011}}_{u^{2011}} \underbrace{\cos x \, dx}_{du}$$


$$d(\sin x) = \cos x \, dx$$

$$u = \sin x \quad du = \cos x \, dx$$

$$I = \int u^{2011} \, du = \frac{u^{2012}}{2012} + C = \frac{\sin^{2012} x}{2012} + C$$

EXAMPLE 2. Evaluate $I = \int \sin^5 x \, dx = \int \sin^4 x \underbrace{\sin x \, dx}_{-d(\cos x)}$

$I = \int (\underbrace{\sin^2 x}_{\text{use } \sin^2 x + \cos^2 x = 1})^2 \sin x \, dx = \int (1 - \cos^2 x)^2 \sin x \, dx$

$u = \cos x \quad du = -\sin x \, dx$

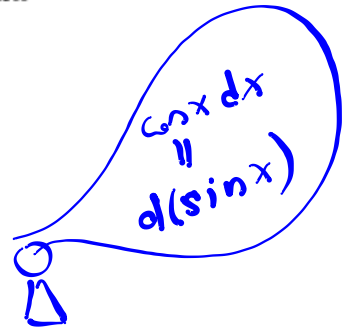
$I = -\int (1 - u^2)^2 \, du = -\int 1 + u^4 - 2u^2 \, du$

$I = -u - \frac{u^5}{5} + \frac{2}{3}u^3 + C$

$I = -\cos x - \frac{\cos^5 x}{5} + \frac{2}{3}\cos^3 x + C$

EXAMPLE 3. Evaluate $I = \int \sin^{2010} x \cos^3 x \, dx$

$$I = \int (\sin x)^{2010} \underbrace{\cos^2 x}_{1 - \sin^2 x} \cos x \, dx$$



$$u = \sin x \quad du = \cos x \, dx$$

$$I = \int u^{2010} (1 - u^2) \, du = \int u^{2010} - u^{2012} \, du$$

$$I = \frac{u^{2011}}{2011} - \frac{u^{2013}}{2013} + C$$

$$I = \frac{\sin^{2011} x}{2011} - \frac{\sin^{2013} x}{2013} + C$$

RULE: How to evaluate $\int \sin^n x \cos^m x \, dx$

1. If n is odd use substitution $u = \cos x$ (Strip out one sine and convert the rest to cosine.)
2. If m is odd use substitution $u = \sin x$ (Strip out one cosine and convert the rest to sine.)
3. If both n and m are odd use 1 or 2.
4. If both n and m are even, use the ~~half~~^{double}-angle identities:

EXAMPLE 4. Evaluate $I = \int_0^{\pi/2} \sin^2 x \, dx$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$I = \frac{1}{2} \int_0^{\pi/2} 1 - \cos(2x) \, dx = \frac{1}{2} \left(x - \frac{\sin 2x}{2} \right) \Big|_0^{\pi/2} =$$

$$= \frac{1}{2} \left(\frac{\pi}{2} - \frac{\sin \pi}{2} - \left(0 - \frac{\sin 0}{2} \right) \right) = \frac{\pi}{4}$$

Note that it immediately implies that

$$\int_0^{\pi/2} \cos^2 x \, dx = \frac{\pi}{4}$$

EXAMPLE 5. Evaluate $I = \int \sin^2 x \cos^2 x dx$

$$2 \sin x \cos x = \sin 2x$$

$$\sin x \cos x = \frac{1}{2} \sin 2x$$

$$I = \int (\sin x \cos x)^2 dx = \int \left(\frac{1}{2} \sin 2x\right)^2 dx$$

$$I = \frac{1}{4} \int \sin^2 2x dx = \text{use } \sin^2 2x = \frac{1}{2}(1 - \cos 4x)$$

$$I = \frac{1}{4} \int \frac{1}{2} (1 - \cos 4x) dx = \frac{1}{8} \int 1 - \cos 4x dx$$

$$I = \frac{1}{8} \left(x - \frac{\sin 4x}{4} \right) + C$$

2nd way

$$I = \int \sin^2 x \cos^2 x dx = \int \sin^2 x (1 - \sin^2 x) dx$$

$$I = \int (\sin^2 x - \sin^4 x) dx = \int \frac{1}{2} (1 - \cos 2x) - \left(\frac{1}{2} (1 - \cos 2x) \right)^2 dx$$

$$I = \frac{1}{2} \left(x - \frac{\sin 2x}{2} \right) - \frac{1}{4} \int (1 - 2\cos 2x + \cos^2 2x) dx$$

$$I = \frac{1}{2} \left(x - \frac{\sin 2x}{2} \right) - \frac{1}{4} \left[x - \sin 2x + \int \frac{1 + \cos 2x}{2} dx \right]$$

RULE: How to evaluate $\int \sec^n x \tan^m x \, dx$

1. *If n is even* use formula $\sec^2 x = 1 + \tan^2 x$ and substitution

$$d(\tan x) = \sec^2 x \, dx$$

$$u = \tan x \quad \Rightarrow \quad du = \sec^2 x \, dx.$$

2. *If m is odd* use formula $\tan^2 x = \sec^2 x - 1$ and substitution

$$d(\sec x) = \sec x \tan x \, dx$$

$$u = \sec x \quad \Rightarrow \quad du = \sec x \tan x \, dx$$

REMARK 6. Integral $\int \csc^n x \cot^m x \, dx$ can be found by similar methods because of the identity $1 + \cot^2 x = \csc^2 x$

EXAMPLE 7. Evaluate $I = \int \sec^{2011} x \tan^5 x \, dx$

$$I = \int (\sec x)^{2010} \underbrace{\tan^4 x}_{(\tan^2 x)^2 = (\sec^2 x - 1)^2} \underbrace{\sec x \tan x \, dx}_{d(\sec x)} = d(\sec x)$$

$u = \sec x$ $du = \sec x \tan x \, dx$

$$I = \int u^{2010} (u^2 - 1)^2 \, du = \int u^{2010} (u^4 - 2u^2 + 1) \, du$$

$$I = \int u^{2014} - 2u^{2012} + u^{2010} \, du$$

$$I = \frac{\sec^{2015} x}{2015} - \frac{2 \sec^{2013} x}{2013} + \frac{\sec^{2011} x}{2011} + C$$

EXAMPLE 8. Evaluate $I = \int \frac{\sin^{10} x}{\cos^{14} x} dx$

$$I = \int \frac{\sin^{10} x}{\cos^{10} x} \cdot \frac{1}{\cos^4 x} dx = \int (\tan x)^{10} (\sec x)^4 dx$$

$$I = \int (\tan x)^{10} \underbrace{\sec^2 x}_{1 + \tan^2 x} \underbrace{\sec^2 x dx}_{d(\tan x)}$$

$$u = \tan x \quad \Rightarrow \quad du = \sec^2 x dx$$

$$I = \int u^{10} (1 + u^2) du = \int u^{10} + u^{12} du$$

$$I = \frac{\tan^{11} x}{11} + \frac{\tan^{13} x}{13} + C$$

RULE: How to evaluate $\int \sin(Ax) \cos(Bx) dx$, $\int \sin(Ax) \sin(Bx) dx$, $\int \cos(Ax) \cos(Bx) dx$
Use the following identities:

$$\sin A \cos B = \frac{1}{2} (\sin(A - B) + \sin(A + B))$$

$$\sin A \sin B = \frac{1}{2} (\cos(A - B) - \cos(A + B))$$

$$\cos A \cos B = \frac{1}{2} (\cos(A - B) + \cos(A + B))$$

EXAMPLE 9. Evaluate $I = \int \cos(25x) \cos(4x) dx$

$$I = \int \left(\frac{1}{2} (\cos(25-4)x + \cos(25+4)x) \right) dx$$

$$I = \frac{1}{2} \int (\cos(21x) + \cos(29x)) dx$$

$$I = \frac{1}{2} \left(\frac{\sin(21x)}{21} + \frac{\sin(29x)}{29} \right) + C$$