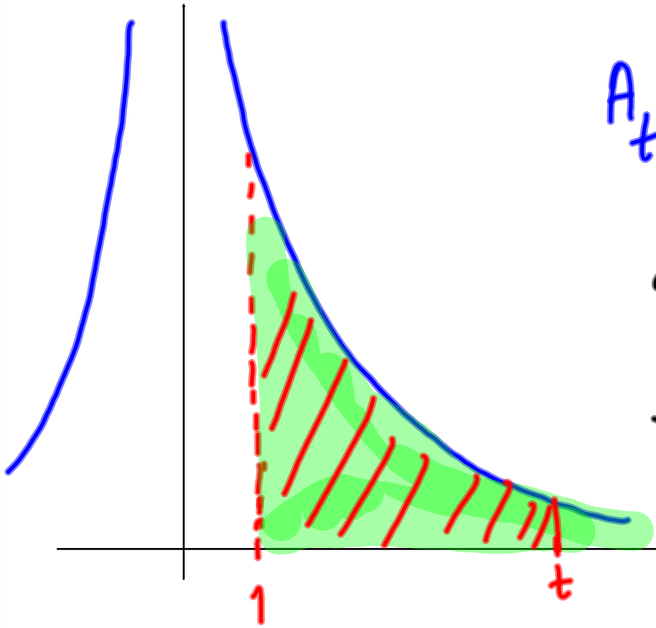
8.9: Improper Integrals

TYPE I: Infinite Interval and Continuous Integrand

EXAMPLE 1. *Evaluate* $\int_1^{\infty} \frac{1}{x^2}\,\mathrm{d}x$

- What is the area, A, under the curve $y = \frac{1}{x^2}$ on $[1, \infty)$ is?
- What is the area, A_t, under the curve $y = \frac{1}{x^2}$ on $[1, t)$, $t > 1$, is?



$$A_t = \int_1^t \frac{dx}{x^2} = -\frac{1}{x}\Big|_1^t = -\left(\frac{1}{t} - 1\right) = 1 - \frac{1}{t}$$

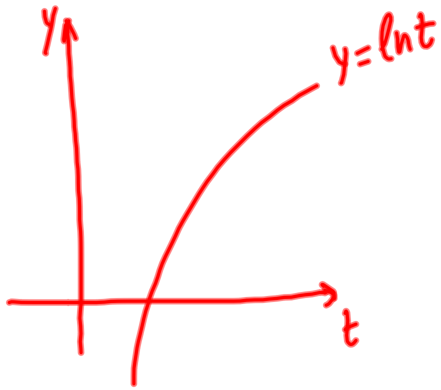
$$\int_1^{\infty} \frac{dx}{x^2} = \lim_{t\to\infty} A_t = 1$$ convergent

improper integral

REMARK 2. Not all areas on an unbounded interval will yield finite areas.

DEFINITION 3. *An improper integral is called* **convergent** *if the associated limit exists and is a finite number. An improper integral is called* **divergent** *if the associated limit does not exist or is* $-\infty$, *or* ∞.

EXAMPLE 4. *Evaluate* $\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{dx}{x}$



$$= \lim_{t\to\infty} \ln|x|\Big|_1^t = \lim_{t\to\infty} \ln t - \underbrace{\ln 1}_{=0}$$

$$= \lim_{t\to\infty} \ln t = \infty$$

divergent

$$\boxed{\text{FACT: If } a>0 \text{ then } \int_a^\infty \frac{1}{x^p}\,dx \text{ is convergent as } p>1 \text{ and divergent as } p\le 1.}$$

$\int_3^\infty \frac{dx}{x^3}$	$\int_4^\infty \frac{dx}{\sqrt{x}}$	$\int_{3/2}^\infty x^{-\frac{3}{2}}\,dx$	$\int_5^\infty x^3\,dx$
$p=3$	$p=\frac{1}{2}$	$p=3/2$	$p=-3$
conv.	diverg.	conv.	divergent

How to deal with Type I Improper Integrals:

- If $\int_a^t f(x)\,\mathrm{d}x$ exists for every $t \geq a$ then

$$\int_a^{\infty} f(x)\,\mathrm{d}x = \lim_{t\to\infty} \int_a^t f(x)\,\mathrm{d}x$$

provided the limit exists and finite.

- If $\int_t^b f(x)\,\mathrm{d}x$ exists for every $t \leq b$ then

$$\int_{-\infty}^{b} f(x)\,\mathrm{d}x = \lim_{t\to-\infty} \int_t^b f(x)\,\mathrm{d}x$$

provided the limit exists and finite.

- If $\int_{-\infty}^c f(x)\,\mathrm{d}x$ and $\int_c^{\infty} f(x)\,\mathrm{d}x$ are BOTH convergent then

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_{-\infty}^{c} f(x)\,\mathrm{d}x + \int_c^{\infty} f(x)\,\mathrm{d}x$$

where c is any number.

EXAMPLE 5. *Evaluate* $I = \int_{-\infty}^{0} \frac{1}{\underbrace{\sqrt{20-x}}_{\text{continuous}}} \mathrm{d}x$

$$I = \lim_{t\to-\infty} \int_t^0 \frac{dx}{\sqrt{20-x}}$$

$$= \lim_{t\to-\infty} -2\sqrt{20-x}\Big|_t^0$$

$$= -2 \lim_{t\to\infty} \left(\sqrt{20} - \sqrt{20-t}\right)$$

$$= -2\sqrt{20} + 2 \lim_{t\to-\infty} \sqrt{20-t} = \infty$$

divergent

$$\int \frac{dx}{\sqrt{20-x}} = -\int u^{-\frac{1}{2}} dx$$

$u = 20 - x$

$du = -dx$

$$= -2\sqrt{u} = -2\sqrt{20-x}$$

Additional thoughts

$$I = \int_{20}^{\infty} \frac{du}{\sqrt{u}} \quad \text{divergent}$$

$p = \frac{1}{2}$

EXAMPLE 6. *Evaluate* $I = \int_{-\infty}^{\infty} \underbrace{xe^{-x^2}}_{\text{continuous}}\, dx$

$$I = \underbrace{\int_{-\infty}^{0} x e^{-x^2} dx}_{I_1} + \underbrace{\int_{0}^{\infty} x e^{-x^2} dx}_{I_2}$$

$$\int x e^{-x^2} dx \qquad u = -x^2 \qquad du = -2x\,dx$$

$$= -\frac{1}{2}\int e^u du = -\frac{1}{2} e^u = -\frac{1}{2} e^{-x^2}$$

$$I_1 = \lim_{t\to-\infty} \int_t^0 x e^{-x^2} dx = -\frac{1}{2} \lim_{t\to-\infty} e^{-x^2}\Big|_t^0$$

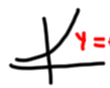
$$= -\frac{1}{2} \lim_{t\to-\infty} \left(e^0 - e^{-t^2}\right) = -\frac{1}{2} + \frac{1}{2} \lim_{t\to-\infty} e^{-t^2}$$

$$= -\frac{1}{2} + \frac{1}{2}\cdot 0 = \boxed{-\frac{1}{2}}$$

To find $\lim_{t\to\mp\infty} e^{-t^2}$

$$x = -t^2 \xrightarrow[t\to\mp\infty]{} -\infty$$

$$\lim_{x\to-\infty} e^x dx = 0$$



$$I_2 = \lim_{t\to\infty} \int_0^t x e^{-x^2} dx = -\frac{1}{2} \lim_{t\to\infty} e^{-x^2}\Big|_0^t =$$

$$= -\frac{1}{2}\left[\lim_{t\to\infty} e^{-t^2} - e^0\right] = -\frac{1}{2}(0-1) = \frac{1}{2}$$

$$I = -\frac{1}{2} + \frac{1}{2} = \boxed{0}$$

convergent

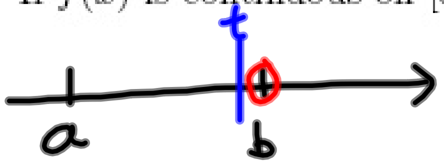
EXAMPLE 7. *Evaluate* $I = \int_{-2}^{\infty} \underbrace{\sin x}_{\text{cont.}} \, dx = \lim_{t\to\infty} \int_{-2}^{t} \sin x \, dx =$

$$= -\lim_{t\to\infty} \cos t \Big|_{-2}^{t} = -\left[\lim_{t\to\infty} \cos t - \cos(-2)\right]$$

$$= \cos 2 - \underbrace{\lim_{t\to\infty} \cos t}_{\text{DNE}} \quad \text{divergent}$$

TYPE II: Discontinuous Integrand and Finite Interval

- If $f(x)$ is continuous on $[a, b)$ and not continuous at $x = b$ then



$$\int_a^b f(x)\,\mathrm{d}x = \lim_{t \to b^-} \int_a^t f(x)\,\mathrm{d}x$$

if the limit exists and finite.

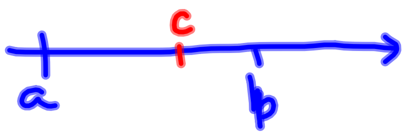
- If $f(x)$ is continuous on $(a, b]$ and not continuous at $x = a$ then

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{t \to a^+} \int_t^b f(x)\,\mathrm{d}x$$

if the limit exists and finite.

- If $f(x)$ is continuous on $[a, c)$ and $(c, b]$ and not continuous at $x = c$, and the integrals $\int_a^c f(x)\,\mathrm{d}x$ and $\int_c^b f(x)\,\mathrm{d}x$ are both convergent then

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x.$$



$$\int_{-2}^{1} \frac{dx}{x^2}$$

EXAMPLE 8. *Evaluate* $I = \int_0^{10} \frac{1}{\sqrt{10-x}}\,dx$

$$I = \lim_{t\to 10^-} \int_0^t \frac{dx}{\sqrt{10-x}} = \lim_{t\to 10^-} -2\sqrt{10-x}\Big|_0^t$$

$x=10$ is not in domain of $y = \frac{1}{\sqrt{10-x}}$ (i.e. the integrand is not cont. at $x=10$)

$$= -2\left(\lim_{t\to 10^-} \sqrt{10-t} - \sqrt{10}\right) = -2\left(\sqrt{10-10} - \sqrt{10}\right) = 2\sqrt{10}$$

I is convergent

EXAMPLE 9. *Evaluate* $I = \int_0^1 \ln x\,dx$

$x=0$ is not in domain of $y=\ln x$

$$I = \lim_{t\to 0^+} \int_t^1 \ln x\,dx$$

$\int \ln x\,dx$

Integr. by parts

$u = \ln x \qquad dv = dx$

$du = \frac{dx}{x} \qquad v = x$

$\int \ln x\,dx = x\ln x - x$

$$= \lim_{t\to 0^+} \left(x\ln x - x\right)\Big|_t^1$$

$$= \lim_{t\to 0^+} \left(1\cdot\ln 1 - 1 - t\ln t + t\right) = -1 - \lim_{t\to 0^+} t\ln t$$

$0\cdot\infty$

$$= -1 - \lim_{t\to 0^+} \frac{(\ln t)'}{\left(\frac{1}{t}\right)'} = \text{L'Hôpital}$$

$\frac{\infty}{\infty}$

$$= -1 - \lim_{t\to 0^+} \frac{\frac{1}{t}}{-1/t^2} = 1 + \lim_{t\to 0^+} t = 1+0 = -1$$

I is convergent

EXAMPLE 10. *Evaluate* $I = \int_{-2}^{3} \frac{1}{x^3}\, dx$

$y = \frac{1}{x^3}$ is discont. at $x=0$

$$I = \int_{-2}^{0} \frac{dx}{x^3} + \int_{0}^{3} \frac{dx}{x^3}$$

$$I = \underbrace{\lim_{t\to 0^-} \int_{-2}^{t} \frac{dx}{x^3}}_{} + \underbrace{\lim_{t\to 0^+} \int_{t}^{3} \frac{dx}{x^3}}_{}$$

no need to work it if the first one is divergent

$$\lim_{t\to 0^-} -\frac{1}{2x^2}\Bigg|_{-2}^{t}$$

$$-\frac{1}{2}\lim_{t\to 0^-}\left(\frac{1}{t^2} - \frac{1}{4}\right) = -\infty$$

divergent

$\Rightarrow I$ is divergent

Now we consider an integral involving both of these cases.

EXAMPLE 11. *Evaluate* $I = \int_0^\infty \frac{1}{x^2}\,dx$

infinite interval + discont. integrand

$$I = \underbrace{\int_0^1 \frac{dx}{x^2}}_{\text{TYPE II}} + \underbrace{\int_1^\infty \frac{dx}{x^2}}_{\text{TYPE I}}$$

conv. $p = 2 > 1 \Rightarrow$ convergent

$$\int_0^1 \frac{dx}{x^2} = \lim_{t\to 0^+} \int_t^1 \frac{dx}{x^2} = -\lim_{t\to 0^+} \frac{1}{x}\Big|_t^1$$

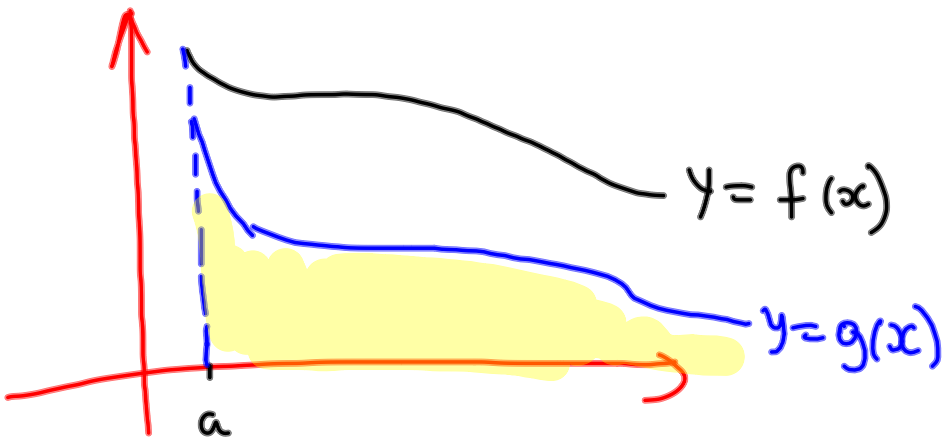
$$= -\lim_{t\to 0^+}\left(1 - \frac{1}{t}\right) = \infty \text{ divergent}$$

I is divergent

Related to TYPE I improper integrals

Comparison Theorem: Suppose $f(x)$ and $g(x)$ are continuous functions s.t $f(x) \geq g(x) \geq 0$ for $x \geq a$. Then

1. if $\int_a^\infty f(x)\,dx$ is convergent then $\int_a^\infty g(x)\,dx$ is convergent;
2. if $\int_a^\infty g(x)\,dx$ is divergent then $\int_a^\infty f(x)\,dx$ is divergent.



$$\int_a^\infty f(x)dx \text{ conv.} \Rightarrow \int_a^\infty g(x)dx \text{ conv.}$$

$$\int_a^\infty g(x)dx \text{ diverg.} \Rightarrow \int_a^\infty f(x)dx \text{ diverg.}$$

EXAMPLE 12. *Determine whether the following integrals are convergent or divergent.*

(a) $I = \int_1^{\infty} \frac{\sin^2 x}{x^2}\,\mathrm{d}x$

We do not know how to find antiderivative of the integrand.

We know that $0 \le \sin^2 x \le 1$

$$0 \le \underbrace{\frac{\sin^2 x}{x^2}}_{g(x)} \le \underbrace{\frac{1}{x^2}}_{f(x)}$$

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{dx}{x^2} \text{ converges } (p = 2 > 1)$$

By Comparison Theorem I is convergent

(b) $I = \int_1^\infty \frac{\sqrt{1+\sqrt{x}}}{\sqrt[3]{x}}\,dx$

$1+\sqrt{x} \geq 1$ (for all x)

$$\underbrace{\frac{\sqrt{1+\sqrt{x}}}{\sqrt[3]{x}}}_{f(x)} \geq \underbrace{\frac{1}{\sqrt[3]{x}}}_{g(x)} \geq 0$$

$$\int_1^\infty \frac{1}{\sqrt[3]{x}}\,dx = \int_1^\infty \frac{dx}{x^{1/3}}$$ divergent $\left(p = \frac{1}{3}\right)$

By Comp. Theorem I is divergent.

(c) $I = \int_1^\infty \frac{1}{x+e^{2x}}\,\mathrm{d}x$

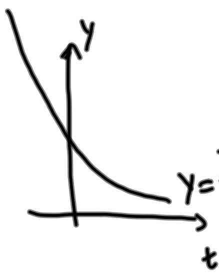
$$\underbrace{x}_{x\geq 1} + \underbrace{e^{2x}}_{>0} \geq x$$

$$0 \leq \frac{1}{x+e^{2x}} \leq \frac{1}{x}$$

$\int_1^\infty \frac{dx}{x}$ diverg.

$\Downarrow$

?



$$\underbrace{x}_{x\geq 0} + e^{2x} \geq e^{2x}$$

$$0 \leq \frac{1}{x+e^{2x}} \leq \frac{1}{e^{2x}}$$

Check if $\int_1^\infty \frac{dx}{e^{2x}}$ convergent

$$\int_1^\infty \frac{dx}{e^{2x}} = \lim_{t\to\infty} \int_1^t e^{-2x}\,dx$$

$$= -\frac{1}{2} \lim_{t\to\infty} e^{-2x}\Big|_1^t$$

$$= -\frac{1}{2}\left(\lim_{t\to\infty} e^{-2t} - e^{-2}\right)$$

$$= -\frac{1}{2}\left(0 - e^{-2}\right) = -\frac{1}{2}e^{-2}$$

convergent

By Comparison Theorem

I is convergent.