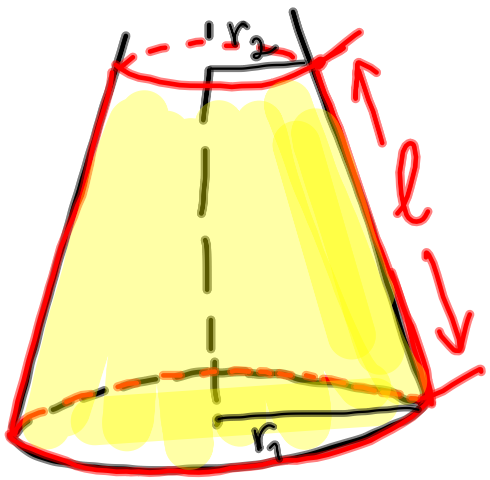
9.4: Area of a Surface of Revolution

Question: What is the area of the band (or frustum of a cone?)

Answer:

$$r = \frac{r_1 + r_2}{2}$$

$$SA = 2\pi r \ell$$

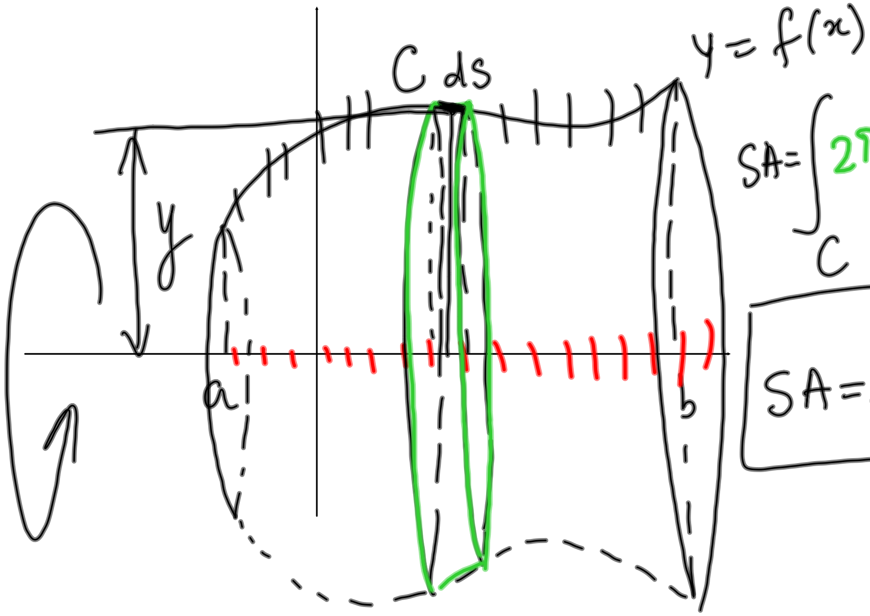


PROBLEM: Find the surface area of solid obtained by rotating the curve

$$C: \quad y = f(x), a \leq x \leq b$$

about the x-axis. (Assume that f is nonnegative and continuous on $[a, b]$.)

Solution: Approximate the surface area by areas of approximating bands:



$$SA = \int_C 2\pi(\text{radius})\, ds$$

$$\boxed{SA = 2\pi \int_C y\, ds}$$

<u>Note</u> If we rotate a curve C about the y-axis

$$SA = 2\pi \int_C x\, ds$$

EXAMPLE 1. *Find the surface area of the solid obtained by rotating the curve*

$$C: x = R\cos t, \quad y = R\sin t, \quad \boxed{0 \le t \le \pi} \; (R > 0)$$

about the x-axis.

$$SA = 2\pi \int_C (\text{radius})\, dS = 2\pi \int_C y\, ds = 2\pi \int_0^{\pi} R\sin t \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

$$SA = 2\pi R \int_0^{\pi} \sin t \sqrt{(-R\sin t)^2 + (R\cos t)^2}\, dt$$

$$SA = 2\pi R \int_0^{\pi} \sin t \sqrt{R^2 (\underbrace{\sin^2 t + \cos^2 t}_{=1})}\, dt$$

$$SA = 2\pi R^2 \underbrace{\int_0^{\pi} \sin t\, dt}_{=2} = 4\pi R^2$$

EXAMPLE 2. *Find the surface area of the solid obtained by revolving the curve given by*

$$\boxed{y^2 = 4x + 4,} \quad 2 \le y \le 6,$$

about the x-axis.

$$SA = 2\pi \int_C (\text{radius})\, ds = 2\pi \int_C y\, ds = 2\pi \int_2^6 y \sqrt{1 + [x'(y)]^2}\, dy$$

$$4x = y^2 - 4$$

$$x = \frac{y^2 - 4}{4} \implies x'(y) = \frac{y}{2}$$

$$SA = 2\pi \int_2^6 y \sqrt{1 + \frac{y^2}{4}}\, dy = 2\pi \underbrace{\int_2^6 \frac{y}{2} \sqrt{4 + y^2}\, dy}_{\text{use u-sub. to get}}$$

$$u = 4 + y^2$$

$$SA = \frac{16\sqrt{2}}{3}\left(5\sqrt{5} - 1\right)$$

EXAMPLE 3. *Determine the surface area of the solid obtained by revolving the curve given by*

$$y = \sqrt[3]{x}, \quad 1 \leq x \leq 8,$$

about the y-axis.

$$SA = 2\pi \int_C (\text{radius})\, ds = 2\pi \int_C x\, ds = 2\pi \int_1^8 x\sqrt{1+(y'(x))^2}\, dx$$

$$y = \sqrt[3]{x} = x^{\frac{1}{3}} \Rightarrow y' = \frac{1}{3x^{2/3}}$$

$$\sqrt{1+(y')^2} = \sqrt{1+\frac{1}{9x^{4/3}}} = \sqrt{\frac{9x^{4/3}+1}{9x^{4/3}}} = \frac{\sqrt{9x^{4/3}+1}}{3x^{2/3}}$$

$$SA = 2\pi \int_1^8 x \cdot \frac{\sqrt{9x^{4/3}+1}}{3x^{2/3}}\, dx$$

$$SA = \frac{2\pi}{3} \int_1^8 x^{\frac{1}{3}} \sqrt{\underbrace{9x^{4/3}+1}_{u}}\, dx = \ldots = \frac{2\pi}{3}\int_{10}^{145} \sqrt{u}\, \frac{du}{12}$$

$$= \frac{\pi}{27}\left(145^{3/2} - 10^{3/2}\right)$$

$$\approx 199.48$$

EXAMPLE 4. *Determine the surface area of the solid obtained by rotating*

$$y = \sqrt{9 - x^2}, \quad \underline{|x| \le 2,} \quad \Rightarrow -2 \le x \le 2$$

about the x-axis.

$$SA = 2\pi \int_C (\text{radius})\, ds = 2\pi \int_C y\, ds$$

$$SA = 2\pi \int_{-2}^{2} \sqrt{9-x^2}\, \sqrt{1+[y'(x)]^2}\, dx$$

$$y'(x) = \frac{d}{dx}\left(\sqrt{9-x^2}\right) = \frac{-2x}{2\sqrt{9-x^2}} = -\frac{x}{\sqrt{9-x^2}}$$

$$\sqrt{1+[y'(x)]^2} = \sqrt{1 + \frac{x^2}{9-x^2}} = \sqrt{\frac{9-x^2+x^2}{9-x^2}} = \frac{3}{\sqrt{9-x^2}}$$

$$SA = 2\pi \int_{-2}^{2} \cancel{\sqrt{9-x^2}}\, \frac{3}{\cancel{\sqrt{9-x^2}}}\, dx = 6\pi \int_{-2}^{2} dx = \boxed{24\pi}$$