10. Consider $y' = ay + b$, where a and b are arbitrary real constants.

(a) Find general solution.

(b) Find solution satisfying the initial condition $y(0) = y_0$.

we got it on wednesday

(a) answer: $y(t) = Ce^{at} - \frac{b}{a}$

(b) answer: $y(t) = \left(y_0 + \frac{b}{a}\right)e^{at} - \frac{b}{a}$

11. Solution that doesn't change with time (i.e. $y(t) = y_0$ for any t) is called an **equilibrium solution**. The corresponding y_0 is called an **equilibrium or stationary point.**

$y_0 + \frac{b}{a} = 0 \Rightarrow \underline{y_0 = -\frac{b}{a}}$

Then $\underbrace{y(t) = -\frac{b}{a}}_{\text{equilibrium solution for } y' = ay + b}$ for all t

$y' = 0$

$\Rightarrow ay + b = 0 \Rightarrow y = -\frac{b}{a}$

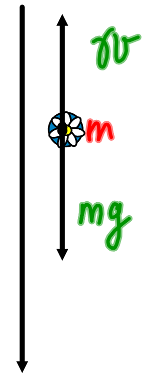
$y_e = -\frac{b}{a}$

eq. point

12. An ODE that describes some physical process is called **mathematical model.**

13. **A model of free-fall motion retarded by air resistance.**

Suppose that an object with mass m falls through air toward Earth. Assume that the only forces acting on the object are gravity and air resistance (=drag force). Denote by γ the drag coefficient. Assume drag force is proportional to velocity $v(t)$



(a) Formulate an ODE that describes the motion

(b) Find general solution of the ODE obtained in the previous item

(c) Find a particular solution of the ODE obtained in (b) satisfying the initial condition $v(0) = v_0$.

(a) net force $= mg - \gamma v$

By 2nd Newton's law net force $= ma$

$$ma = mg - \gamma v$$

$$a = v'$$

$$mv' = mg - \gamma v$$

$$\boxed{v' = -\frac{\gamma}{m} v + g}$$

compare

1st order ODE (Linear with constant coefficients)

see (10)

$$y' = ay + b$$

$$y(t) = Ce^{at} - \frac{b}{a}$$

$$y(t) = \left(y_0 + \frac{b}{a}\right)e^{at} - \frac{b}{a}$$

(b) $y = V$, $a = -\frac{\gamma}{m}$, $b = g$

$$\boxed{v(t) = Ce^{-\frac{\gamma}{m}t} + \frac{mg}{\gamma}}$$

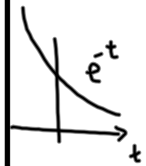
$$\frac{b}{a} = \frac{g}{-\gamma/m} = -\frac{mg}{\gamma}$$

(c) $v(0)=v_0$

By (10)

$$v(t)=\left(v_0-\frac{mg}{\gamma}\right)e^{-\frac{\gamma}{m}t}+\frac{mg}{\gamma}$$

(d) Limiting velocity and Equilibrium



$$\underbrace{\text{Limiting velocity}} = \lim_{t\to+\infty} v(t) = \underbrace{\lim_{t\to+\infty} Ce^{-\frac{\gamma t}{m}}}_{=0} + \frac{mg}{\gamma} = \underbrace{\frac{mg}{\gamma}}_{\text{limiting velocity}}$$

near limiting velocity

$$\frac{dv}{dt} = v' \approx 0$$

$$= -\frac{\gamma}{m}v + g \approx 0$$

$$g \approx \frac{\gamma v}{m} \Rightarrow \underbrace{mg}_{\text{gravity}} \approx \underbrace{\gamma v}_{\text{drag force}}$$

Equilibrium: $v'=0 \Rightarrow -\frac{\gamma}{m}v+g=0 \Rightarrow mg=\gamma v$

Equilibrium corresponds to the perfect balance between gravity and air resistance

2. Separable Equations (section 2.2)

1. Separable ODE is any ODE that is expressible in one of the following forms:

$$y' = f(t)g(y), \qquad p(y)\frac{\mathrm{d}y}{\mathrm{d}t} = q(t).$$

2. Steps to solve a separable ODE:

- Separate variables, i.e. rewrite the given equation in the differential form:

$$\boxed{p(y)\mathrm{d}y = q(t)\mathrm{d}t.} \qquad dy = y'(t)dt \tag{1}$$

- Integrate both sides of (1):

$$\int p(y)\mathrm{d}y = \int q(t)\mathrm{d}t. \tag{2}$$

- If $P(y)$ and $Q(t)$ are antiderivatives of $p(y)$ and $q(t)$ respectively, then the equation $P(y) = Q(t) + C$ will define a family of solutions implicitly. (Note, in some cases it may be possible to solve this equation for y to find the solution explicitly.)

$$P(y) + \cancel{C} = Q(t) + \cancel{C_1} + C$$

3. (a) Find general solution of $ty' = -y$ $\quad (t > 0)$ and sketch some typical integral curves.

(b) Solve IVP: $ty' = -y \quad (t > 0), \quad y(4) = 2.$

(a) separate variables

$$ty' = -y$$

$$t\frac{dy}{dt} = -y$$

$$t\,dy = -y\,dt$$

$$\frac{dy}{y} = -\frac{dt}{t}, \quad y \neq 0$$

Integrate:

$$\int \frac{dy}{y} = -\int \frac{dt}{t}$$

$$\boxed{\ln|y| = -\ln t + C}$$

Solution in implicit form

and $y=0$

get explicit solution

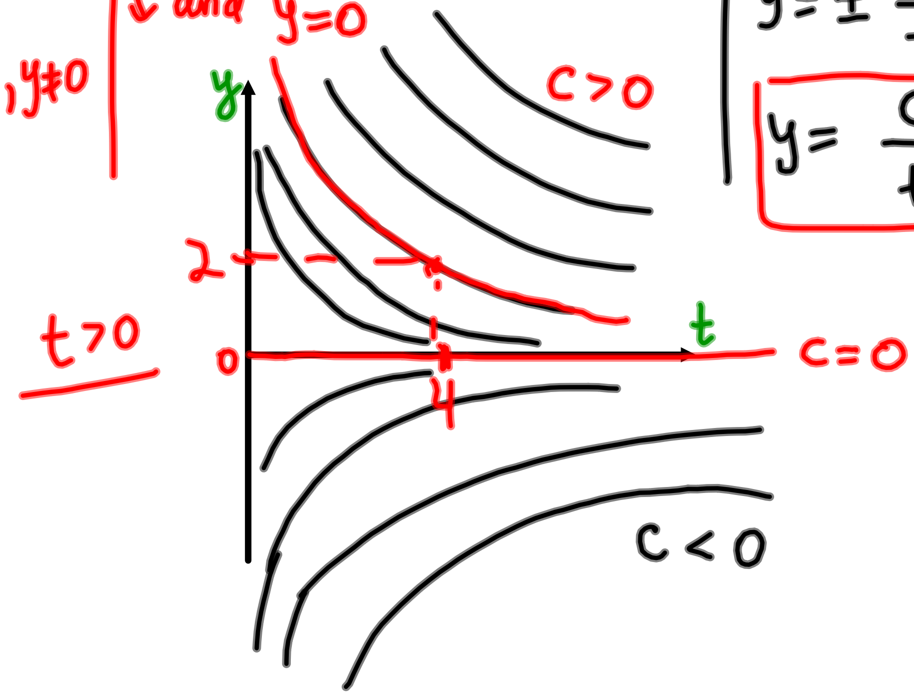
$$|y| = e^{-\ln t + C}$$

$$|y| = e^{-\ln t} \cdot e^{C}$$

$$|y| = C e^{-\ln t}, \quad C > 0$$

$$y = \pm \frac{C}{t}, \quad C > 0$$

$$\boxed{y = \frac{C}{t}}$$ explicit function



(b) $y(4) = 2$

$$y(t) = \frac{C}{t}$$

$$\frac{C}{4} = y(4) = 2$$

$$\Downarrow$$

$$C = 8$$

$$\boxed{y(t) = \frac{8}{t}}$$

5. Separate variables in the following equations:

(a) $$\frac{dy}{dx} = e^{5\sin x - y^3}$$

(b) $$\frac{du}{dt} = 1 + t - u - ut.$$

(a) $$\frac{dy}{dx} = e^{5\sin x} e^{-y^3}$$

$$dy = e^{5\sin x} e^{-y^3} dx$$

$$\frac{dy}{e^{-y^3}} = e^{5\sin x} dx$$

$$e^{y^3} dy = e^{5\sin x} dx$$

(b) $$\frac{du}{dt} = 1 + t - u - ut$$

$$= 1 + t - u(1+t)$$

$$= (1-u)(1+t)$$

$$1 \cdot \quad - u \quad = (1-u)$$

$$\frac{du}{dt} = (1-u)(1+t)$$

$$du = (1-u)(1+t)\,dt$$

$$\frac{du}{1-u} = (1+t)\,dt$$