14: Nonhomogeneous Equations. Method of Undetermined Coefficients (section 3.5)

1. If $y_1(t)$ and $y_2(t)$ are two particular solutions of a second-order *nonhomogeneous* linear ODE,

$$y'' + p(t)y' + q(t)y = g(t), \quad g(t) \not\equiv 0.$$

Then $y_1(t) - y_2(t)$ is a particular solution of the corresponding homogeneous equation,

$$y'' + p(t)y' + q(t)y = 0.$$

$$f' - g' = (f-g)'$$

$$\begin{array}{r} y_1'' + p y_1' + q y_1 = g \\ - \quad y_2'' + p y_2' + q y_2 = g \\ \hline (y_1 - y_2)'' + p(y_1' - y_2') + q(y_1 - y_2) = 0 \end{array}$$

$\Rightarrow y_1 - y_2$ is a solution of homog. eq.

In particular

$y(t)$ general solution of NHLE $-$ $y_p(t)$ particular solution of NHLE $=$ $y_h(t)$ solution of the corresponding HLE

$$\boxed{y(t) = y_p(t) + y_h(t)}$$

2. *THEOREM: Solution of Nonhomogeneous Linear Equation*

Let

$$y'' + p(t)y' + q(t)y = g(t), \quad g(t) \not\equiv 0$$

be a second-order *nonhomogeneous* linear differential equation. If $y_p(t)$ is a particular solution of this equation and $y_h(t)$ is the general solution of the corresponding homogeneous equation,

$$y'' + p(t)y' + q(t)y = 0,$$

then

$$y(t) = y_h(t) + y_p(t)$$

is the general solution of the nonhomogeneous equation.

3. To solve a nonhomogeneous linear ODE:

   Step 1: Find a particular solution of a nonhomogeneous linear ODE. $y_p(t)$

   Step 2: Find general solution of the corresponding homogeneous linear ODE. $y_h(t)$

   Step 3: Add the results of Steps 1&2. $y = y_p + y_h$

4. One solution of $y'' - y = t$ is $y(t) = -t$, as you can verify. What is the general solution?

$y'' - y = 0$

$r^2 - 1 = 0$

$r = \pm 1$

$y_h(t) = c_1 e^{t} + c_2 e^{-t}$

$y(t) = y_p + y_h$

$\boxed{y(t) = -t + c_1 e^{t} + c_2 e^{-t}}$

5. Consider

$$y'' + p(t)y' + q(t)y = g_1(t) + g_2(t) \tag{1}$$

. If $y(t) = y_p(t)$ and $y(t) = Y_p(t)$ are particular solutions of

$$y'' + p(t)y' + q(t)y = g_1(t)$$

and

$$y'' + p(t)y' + q(t)y = g_2(t),$$

respectively, then $y(t) = y_p(t) + Y_p(t)$ is a particular solution of (1).

Method of Undetermined Coefficients

How to find $y_p(t)$?

6. Consider a particular class of nonhomogeneous linear ODE with constant coefficients

$$ay'' + by' + cy = g(t),$$

where a, b, c are real constants and $g(t)$ involves linear combinations, sums and products of

$$t^m, \quad e^{\alpha t}, \quad \sin(\beta t), \quad \cos(\beta t).$$

7. The idea of the Method of Undetermined Coefficients: guess a particular solution $y_p(t)$ using a generalized form of $g(t)$. Then by substitution determine the coefficients for the generalized solution.

8. Find general solution:

(a) $y'' - 3y' + 2y = 4e^{3t}$ $= g(t)$

Homogeneous

$y'' - 3y' + 2y = 0$

$r^2 - 3r + 2 = 0$

$(r-1)(r-2) = 0$

$r_1 = 1, \ r_2 = 2$

$y_h = c_1 e^t + c_2 e^{2t}$

$\boxed{\begin{array}{l} \alpha = 3 \\ \alpha \neq r_{1,2} \\ s = 0 \quad t^s = 1 \end{array}}$

$y_p(t) = \underbrace{A}_{\text{undetermined coefficient}} e^{3t}$

$2 \quad y = 2Ae^{3t}$

$+$

$-3 \quad y' = 3 \cdot 3Ae^{3t}$

$+$

$y'' = 9Ae^{3t}$

$= 0$

$4e^{3t} = 2Ae^{3t}$

$4 = 2A \Rightarrow A = 2$

$y_p(t) = 2e^{3t}$

General solution $\quad y(t) = \underbrace{2e^{3t}}_{y_p} + \underbrace{c_1 e^t + c_2 e^{2t}}_{y_h}$

(b) $y'' - 3y' + 2y = 4e^t$

$y_p(t) = Ate^t$

$y_h = c_1 e^t + c_2 e^{2t}$

$2 \quad y = 2Ate^t$

$-3 \quad y' = -3Ae^t - 3Ate^t$

$y'' = Ae^t + Ae^t + Ate^t$

$= 0$

$4e^t = -3Ae^t + 2Ae^t$

$4e^t = -Ae^t$

$\boxed{A = -4}$

$y_p = -4te^t$

$\boxed{y(t) = -4te^t + c_1 e^t + c_2 e^{2t}}$

$\left.\begin{matrix} \alpha = 1 \\ r_1 = 1, \ r_2 = 2 \end{matrix}\right\} \Rightarrow \alpha = r_1 \quad \boxed{s = 1}$

(c) $y'' + 10y' + 25y = 3e^{-5t}$

$y_p(t) = At^2 e^{-5t}$

$r^2 + 10r + 25 = 0$

$(r+5)^2 = 0$

$r_{1,2} = -5$

$y_h(t) = c_1 e^{-5t} + c_2 t e^{-5t}$

$\left.\begin{matrix} \alpha = -5 \\ r_1 = r_2 = -5 \end{matrix}\right\} \boxed{s = 2}$

$25 \quad y = 25At^2 e^{-5t}$

$10 \quad y' = 2Ate^{-5t} - 5At^2 e^{-5t}$

$y'' = 2Ae^{-5t} - 10Ate^{-5t} - 10Ate^{-5t} + 25At^2 e^{-5t}$

$= 0$

$3e^{-5t} = 2Ae^{-5t}$ $\quad = 0$

$A = \frac{3}{2}$

$y_p(t) = \frac{3}{2} t^2 e^{-5t}$

$\boxed{y(t) = \frac{3}{2} t^2 e^{-5t} + c_1 e^{-5t} + c_2 t e^{-5t}}$

$g(t) = (\text{polynomial}(t))\, e^{\alpha t} \cos\beta t$ (or $\sin\beta t$)

9. **Multiplicity s of a given number $\alpha + i\beta$**

   If $g(t) = (B_0 t^n + B_1 t^{n-1} + \ldots + B_n)e^{\alpha t}, \quad \beta = 0$, then

   - α doesn't coincide with a root of characteristic polynomial $\Rightarrow s = 0$.
   - α coincides with a non repeated root of characteristic polynomial $\Rightarrow s = 1$.
   - α coincides with a repeated root of characteristic polynomial $\Rightarrow s = 2$.

If $g(t) = (B_0 t^n + B_1 t^{n-1} + \ldots + B_1 t^{n-1} + \ldots + B_n)e^{\alpha t}\cos(\beta t)$ or $g(t) = (B_0 t^n + B_1 t^{n-1} + \ldots + B_n)e^{\alpha t}\sin(\beta t)), \quad \beta \neq 0$, then

- $\alpha + i\beta$ doesn't coincide with a root of characteristic polynomial $\Rightarrow s = 0$.
- $\alpha + i\beta$ coincides with a (complex) root of characteristic polynomial $\Rightarrow s = 1$.

10. If $g(t) = (B_0 t^n + B_1 t^{n-1} + \ldots + B_n)e^{\alpha t}\cos(\beta t)$ or $g(t) = (B_0 t^n + B_1 t^{n-1} + \ldots + B_n)e^{\alpha t}\sin(\beta t))$ then we choose a particular solution in the form

$$y_p(t) = t^s(A_0 t^n + A_1 t^{n-1} + \ldots + A_n)e^{\alpha t}\cos(\beta t)$$

or

$$y_p(t) = t^s(A_0 t^n + A_1 t^{n-1} + \ldots + A_n)e^{\alpha t}\sin(\beta t),$$

respectively.

11. Find general solution of

$$y'' + 2y' + 5y = 3\sin 2t$$

12. Determine a suitable choice for y_p: $e^{\alpha t}\cos\beta t \,/\, e^{\alpha t}\sin\beta t$

	$ay'' + by' + cy = g(t)$	r_1, r_2	$\alpha + i\beta$	s	$y_p(t)$
1	$y'' = e^t$	$r_1 = r_2 = 0$	$\alpha = 1$, $\beta = 0$	0	Ae^t $(t^s = 1)$
2	$y'' - 4y' + 4 = e^{2t}$	$r_1 = r_2 = 2$	$2, \beta = 0$	2	At^2e^{2t}
3	$y'' + 2y' + 10y = 6\sin(3t)$	$r_{1,2} = -1 \pm 3i$	$3i_{\alpha=0}$	0	$A\sin 3t + B\cos 3t$
4	$y'' + 2y' + 10y = 6e^{-t}\sin(3t)$	$r_{1,2} = -1 \pm 3i$	$-1+3i$	1	$Ate^{-t}\sin 3t + Bte^{-t}\cos 3t$
5	$y'' + 2y' + 10y = (t^3 - 1)\sin(3t)$	$r_{1,2} = -1 \pm 3i$	$3i$	0	$(A_0t^3 + A_1t^2 + A_2t + A_3)\sin 3t + (B_0t^3 + B_1t^2 + B_2t + B_3)\cos 3t$
6	$y'' + 2y' + 10y = t^2e^{-t}\sin(3t)$	$r_{1,2} = -1 \pm 3i$	$-1+3i$	1	
7	$y'' = 2t - 2013$	$r_1 = r_2 = 0$	0	2	$(At+B)t^2$