# 16: Mechanical and Electrical Vibrations (section 3.7)

Consider linear dynamical system in which mathematical model is the following IVP:

$$au'' + bu' + cu = g(t), \quad u(0) = u_0, \quad u'(0) = v_0.$$

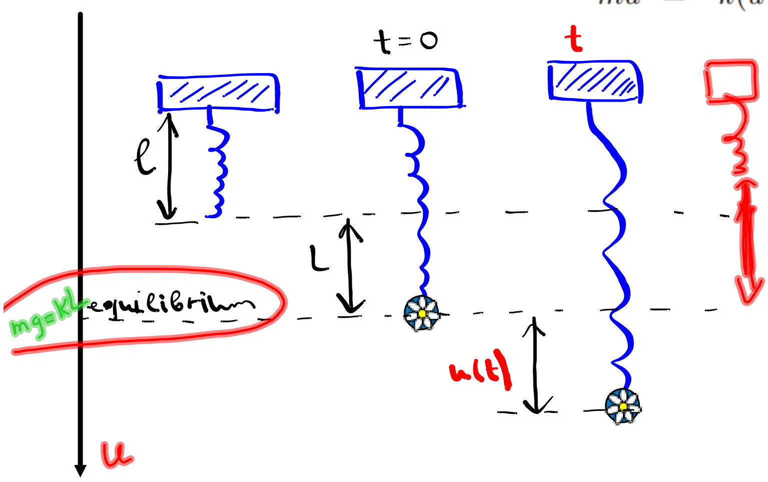
Here $g(t)$ is **forcing function** of the system. A solution $u(t)$ of the DE on an interval containing $t = 0$ that satisfies the initial conditions is called the **response** of the system.

## Spring/mass systems: Free Undamped Vibration (or simple harmonic motion)

1. A flexible spring is suspended vertically from a rigid support and the mass $m$ is attached to the end. By **Hooke's Law**, the spring itself exerts a *restoring force* $F$ opposite to the direction of elongation and proportional to the amount of elongation $L$: $F = -kL$, where $k$ is called the **spring constant**.

2. The mass $m$ stretches the spring by $L$ and attains a position of equilibrium, i.e. weight, $mg$, is balanced by the restoring force:
$$mg - kL = 0.$$

3. If the mass is displaced by an amount $u$ from its equilibrium position, the restoring force is then $-k(u + L)$. Free motion (i.e. no other external/retarding forces acting on the moving mass): use Newton's second Law with the net (or resultant) force:
$$mu'' = -k(u + L) + mg = -ku.$$




ℓ natural spring length

L elongation

u(t) displacement from equilibrium position.

$$mu'' + ku = 0$$

$$mu'' + ku = 0 \Rightarrow u'' + \underbrace{\frac{k}{m}}_{\omega_0^2} u = 0$$

4. DE of Free Undamped Motion:

$$u'' + \omega_0^2 u = 0, \tag{1}$$

where

$$r^2 + \omega_0^2 = 0$$

$$r_{1,2} = \pm i\omega_0$$

$$\omega_0^2 = \frac{k}{m}. \Rightarrow \omega_0 = \sqrt{\frac{k}{m}}$$

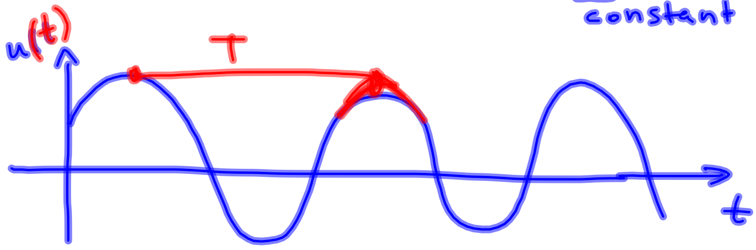
* Initial conditions: $u(0) = u_0$, $\quad u'(0) = v_0$, where $u_0$ is the initial displacement and $v_0$ is the initial velocity. For example, $u_0 < 0$ and $u_1 = 0$ mean that the mass is released from rest from a point $|u_0|$ units above the equilibrium position.

General solution of(1) is

Alternate form

$$u(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) = \underbrace{R}_{\text{constant amplitude}} \cos(\omega_0 t - \delta),$$

where




- $R = \sqrt{C_1^2 + C_2^2}$ is called the **amplitude** of the motion
- $\delta$ is called the **phase**, or phase angle, and measures the displacement of the wave from its normal position corresponding to $\delta = 0$. Recall that

$$\cos\delta = \frac{C_1}{\sqrt{C_1^2 + C_2^2}} = \frac{C_1}{R}, \quad \sin\delta = \frac{C_2}{\sqrt{C_1^2 + C_2^2}} = \frac{C_2}{R}.$$

- $T = \dfrac{2\pi}{\omega_0}$ is the **period** of the motion. The number $T$ is time it takes the mass to execute one cycle of motion (the length of the interval between two successive maxima (or minima) of $u(t)$.)
- $\omega_0 = \sqrt{\dfrac{k}{m}}$ is the **natural frequency** of the system.
- The **frequency of motion** $f = \dfrac{1}{T} = \dfrac{\omega_0}{2\pi}$.

5. A mass weighing 4lb stretches a spring 6 inches. At $t = 0$ the mass released from a point 8 inches below the equilibrium with an upward velocity of 2/3 ft/s. Determine the amplitude of vibrations, phase angle, period, natural frequency of the system and frequency of motion.

no damping

$R$ $\delta$ $T$ $\omega_0$ $f$

$mg = 4$ lb

$L = 6\text{in} = \frac{6}{12} = \frac{1}{2}$ ft

$u(0) = 8\text{ in} = \frac{8}{12} = \frac{2}{3}$ ft

$u'(0) = -\frac{2}{3}$ ft/s

? $R, \delta, T, \omega_0, f$

$mg = 4 \Rightarrow m = \frac{4}{g} = \frac{4}{32} = \frac{1}{8}$ slug

$\underbrace{mg = kL} \Rightarrow k = \frac{mg}{L} = \frac{4}{1/2} = 8$

$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{8}{1/8}} = \sqrt{64} = 8$

$T = \frac{2\pi}{\omega_0} = \frac{2\pi}{8} = \frac{\pi}{4}$

$u'' + \omega_0^2 u = 0 \Rightarrow u'' + 64u = 0$

$r^2 = -64$

$r_{1,2} = \pm 8i$

general solution: $u(t) = C_1 \cos 8t + C_2 \sin 8t$

$u'(t) = -8C_1 \sin 8t + 8C_2 \cos 8t$

To find $C_1, C_2$ use initial conditions

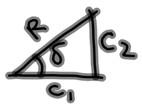
$\frac{2}{3} = u(0) = C_1 \Rightarrow \boxed{C_1 = \frac{2}{3}}$

$-\frac{2}{3} = u'(0) = 8C_2$

$C_2 = -\frac{2}{3 \cdot 8} = \boxed{-\frac{1}{12} = C_2}$

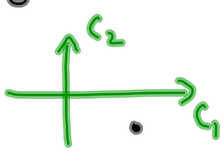
$R = \sqrt{C_1^2 + C_2^2} = \sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{1}{12}\right)^2} = \sqrt{\frac{4}{9} + \frac{1}{144}} = \frac{\sqrt{64+1}}{12} = \frac{\sqrt{65}}{12}$

$R = \frac{\sqrt{65}}{12}$



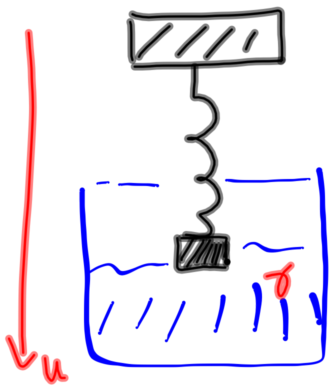

$\tan \delta = \frac{C_2}{C_1} = \frac{-1/12}{2/3} = -\frac{1}{12} \cdot \frac{3}{2} = -\frac{1}{8}$

$$\boxed{\begin{array}{l} \delta = \arctan\left(-\frac{1}{8}\right) \\ \frac{3\pi}{2} < \delta < 2\pi \end{array}}$$



## Spring/mass systems: Free Damped Vibrations.

6. Assume that the mass is suspended in a viscous medium or connected to a dashpot damping device. Dampers work to counteract any movement: damping force $= -\gamma v = -\gamma u'$, where $\gamma$ is a positive damping constant.




Damping force is proportional to instanteneous velocity

$$F_d = -\gamma v = -\gamma u'$$

$$mu'' = \underbrace{mg - k(L+u)}_{-ku} - \gamma u'$$

$$mu'' + \gamma u' + ku = 0$$

7. DE of Free Damped Motion:

$$mu'' + \gamma u' + ku = 0. \tag{2}$$

8. Discriminant of the characteristic equation $mr^2 + \gamma r + k = 0$ is

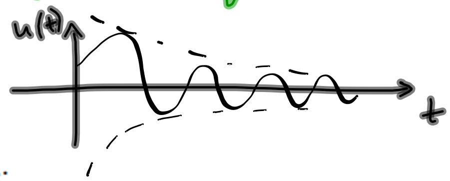
$$D = \gamma^2 - 4mk.$$

CASE 1: (Underdamping) $D < 0$, i.e. the roots are complex conjugate:

$$r_{1,2} = \underbrace{-\frac{\gamma}{2m}}_{\lambda} \pm i \underbrace{\sqrt{\omega_0^2 - \underbrace{\frac{\gamma^2}{4m^2}}_{\lambda^2}}}_{\mu} =: \lambda + i\mu$$

$\lambda = -\dfrac{\gamma}{2m} < 0$

$u(t) = e^{\lambda t}(C_1 \cos \mu t + C_2 \sin \mu t)$




General solution of (2) is not periodic:

$$u(t) = C_1 e^{-\lambda t}\cos(\mu t) + C_2 e^{-\lambda t}\sin(\mu t) = Re^{-\lambda t}\cos(\mu t - \delta),$$

where

- $Re^{-\lambda t}$ is **damped amplitude** of vibrations
- $\mu = \sqrt{\omega_0^2 - \frac{\gamma^2}{4m^2}} = \sqrt{\omega_0^2 - \lambda^2}$ is the **quasi frequency**
- $T_d = \frac{2\pi}{\mu} = \frac{2\pi}{\sqrt{\omega_0^2 - \lambda^2}}$ is the **quasi period**, i.e. the time interval between two successive maxima of $u(t)$.

Note that as $\gamma$ increases, the quasi frequency $\mu$ becomes smaller and the quasi period becomes bigger.

CASE 2: (Critical Damping) $D = 0$ (two repeated (equal) roots) In this case any slight decrease of the damping force would result in oscilatory motion. The general solution of (2) is

$$x(t) = C_1 e^{\lambda t} + C_2 t e^{\lambda t} = \underbrace{e^{-\lambda t}(C_1 + C_2 t)}_{\text{not periodic}}.$$

$D = \gamma^2 - 4mk = 0$

$\Rightarrow$

$\gamma_{critical} = 2\sqrt{mk}$

If $\gamma > \gamma_{critical}$

$\Downarrow$

$D > 0$

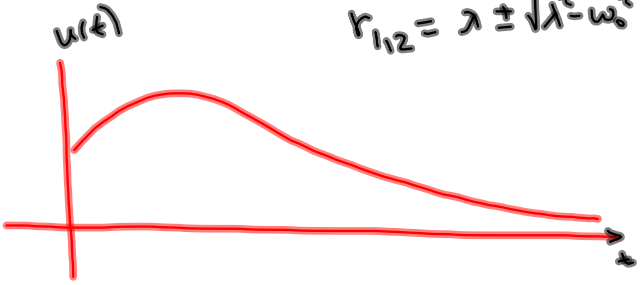
The nature of the general solution $u(t)$ changes from periodic (oscilations) to non-periodic (no oscilations)

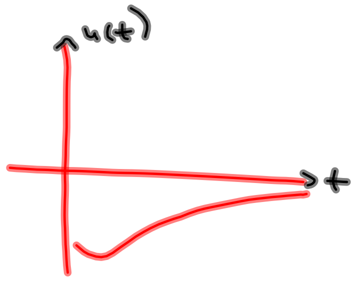
CASE 3: (Overdamping) $D > 0$ (two distinct real roots)In this case there are no oscillation.
The general solution of (2) has no more one zero:

$\gamma > \gamma_{crit}$

$$x(t) = e^{\lambda t}(C_1 e^{\sqrt{\lambda^2-\omega_0^2}t} + C_2 e^{-\sqrt{\lambda^2-\omega_0^2}t}).$$

$r_{1,2} = \lambda \pm \sqrt{\lambda^2 - \omega_0^2}$



**LRC electrical circuit**

9. If $Q$ is the charge at time $t$ in an electrical closed circuit with inductance $L$, resistance $R$, and capacitance $C$, then by Kirchhoff's Second Law (from Physics) the impressed voltage $E(t)$ is equal to the sum of the voltage drops in the rest of the circuit

$$E(t) = IR + \frac{Q}{C} + LI'(t).$$

By substitution $I = Q'$ we get

$$LQ'' + RQ' + \frac{1}{C}Q = E(t).$$

Analogy between electrical and mechanical quantities:

| | |
|---|---|
| Charge $Q$ | Position $u$ |
| Inductance $L$ | mass $m$ |
| Resistance $R$ | Damping constant $\gamma$ |
| Inverse capacitance $1/C$ | Spring constant $k$ |
| Impressed voltage $E(t)$ (electromotive force) | External force $F(t)$ |