19: Solution of Initial Value Problems (sec. 6.2)

1. INVERSE LAPLACE TRANSFORMS: If $F(s)$ represents the Laplace Transform of $f(t)$, i.e. $\mathcal{L}\{f(t)\} = F(s)$, then we say that $f(t)$ is the **inverse Laplace Transform** of $F(s)$ and write
$$f(t) = \mathcal{L}^{-1}\{F(s)\}.$$

2. Some Inverse Transforms:

Transform	**Inverse Transform**
$\mathcal{L}\{1\} = \dfrac{1}{s}$	$\mathcal{L}^{-1}\left\{\dfrac{1}{s}\right\} = 1$
$\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}, \quad n = 1, 2, 3, \ldots$	$\mathcal{L}^{-1}\left\{\dfrac{1}{s^n}\right\} = \dfrac{t^{n-1}}{(n-1)!}, \quad n = 1, 2, 3, \ldots$
$\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$	$\mathcal{L}^{-1}\left\{\dfrac{1}{s-a}\right\} = e^{at}$
$\mathcal{L}\{\sin at\} = \dfrac{a}{s^2+a^2}$	$\mathcal{L}^{-1}\left\{\dfrac{a}{s^2+a^2}\right\} = \sin at$
$\mathcal{L}\{\cos at\} = \dfrac{s}{s^2+a^2}$	$\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+a^2}\right\} = \cos at$
$\mathcal{L}\{t^n e^{\alpha t}\} = \dfrac{n!}{(s-\alpha)^{n+1}}$ $\overset{n \leftrightarrow n-1}{\Longrightarrow}$	$\mathcal{L}^{-1}\left\{\dfrac{1}{(s-\alpha)^n}\right\} = \dfrac{t^{n-1}e^{\alpha t}}{(n-1)!}$

See Table on the page 317 in the Textbook (or Appendix 2) for more cases.

3. $\mathcal{L}^{-1}$ is a Linear Transform:

$$\mathcal{L}^{-1}\left\{\alpha f + \beta g\right\} = \alpha\mathcal{L}^{-1}\left\{f\right\} + \beta\mathcal{L}^{-1}\left\{g\right\}.$$

4. Note that it often happens that a function of s under consideration does not match exactly the form of a Laplace Transform $F(s)$ in the table. In this cases you need to "fix up" the function of s. Helpful strategies:

   - multiply/divide by an appropriate constant
   - use termwise division
   - use Partial Fractions (See Appendix 1: *Inverse Laplace transform of rational functions using Partial Fraction Decomposition*)

5. Example. *Evaluate*

(a) $\mathcal{L}^{-1}\left\{\dfrac{2s+3}{s^2+5s+6}\right\}$

see Case 1 in Appendix 1

$$F(s)=\frac{2s+3}{s^2+5s+6}=\underbrace{\frac{2s+3}{(s+3)(s+2)}}=\frac{A}{s+3}+\frac{B}{s+2}=\underbrace{\frac{A(s+2)+B(s+3)}{(s+3)(s+2)}}$$

Equating numerators:

$$2s+3=A(s+2)+B(s+3)$$

$s=-2 \qquad -4+3=0+B \Rightarrow \boxed{B=-1}$

$s=-3 \qquad -6+3=-A+0 \Rightarrow \boxed{A=3}$

By the Table

$$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\}=e^{at}$$

$$\mathcal{L}^{-1}\{F(s)\}=\mathcal{L}^{-1}\left\{\frac{3}{s+3}-\frac{1}{s+2}\right\}=3\,\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}-\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$=\boxed{3e^{-3t}-e^{-2t}}$$

(b) $\mathcal{L}^{-1}\left\{\dfrac{2s^2-3s+5}{(s-3)^2(s+4)}\right\}$ Cases 1 & 2 in Appendix 1

$$F(s)=\frac{2s^2-3s+5}{(s-3)^2(s+4)}=\frac{A}{s-3}+\frac{B}{(s-3)^2}+\frac{C}{s+4}$$

$$=\frac{A(s-3)(s+4)+B(s+4)+C(s-3)^2}{(s-3)^2(s+4)}$$

$$2s^2-3s+5=A(s-3)(s+4)+B(s+4)+C(s-3)^2$$

$s=3 \Rightarrow 2\cdot 9-9+5=0+7B+0 \Rightarrow 14=7B \Rightarrow \boxed{B=2}$

$s=-4 \Rightarrow 2\cdot 16+12+5=0+0+49C \Rightarrow 49=49C \Rightarrow \boxed{C=1}$

$s^2:\quad 2=A+C \Rightarrow A=2-C \Rightarrow \boxed{A=1}$

$$\mathcal{L}^{-1}\{F(s)\}=\underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}}_{e^{3t}}+2\underbrace{\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\}}_{\frac{t\,e^{3t}}{1!}}+\underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}}_{e^{-4t}}$$

$$e^{3t}+2\,\frac{t\,e^{3t}}{1!}+e^{-4t}$$

$$=e^{3t}+2te^{3t}+e^{-4t}$$

(c) $\mathcal{L}^{-1}\left\{\dfrac{3s+5}{s^2+6s+34}\right\}$

We did it on Wednesday

6. Consider the n-th order ODE

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + a_1 y' + a_0 y = g(t)$$

subject to

$$y(0) = \alpha_0, y'(0) = \alpha_1, \dots y^{(n-1)} = \alpha_{n-1}.$$

Note that in the case $n = 2$ know how to solve this IVP using the Method Variation of Parameters and the Method of Undetermined Coefficients (for $g(t) = P_n(t)e^{\alpha t}\cos bt$ or $g(t) = P_n(t)e^{\alpha t}\sin bt$) [2]

7. How to solve the given IVP using Laplace Transform:

Step 1. Apply Laplace Transform to both sides of the given ODE. Use linearity and other Laplace Transform properties together with the initial conditions to we obtain an algebraic equation in the s-domain for $Y(s) = \mathcal{L}\{y(t)\}$ instead of the given ODE in the t-domain.

Step 2. Solve for $Y(s)$ the algebraic equation obtained in Step 1.

Step 3. Find the inverse Laplace Transform of $Y(s)$ to get $y(t)$.

8. EXAMPLE *Solve IVP*

$$y'' - 2y' + 5y = -8e^{-t}, \quad y(0) = 2, \quad y'(0) = 12.$$

Note that we already found that

$$Y(s) = \frac{2s^2 + 10s}{(s+1)\underbrace{(s^2 - 2s + 5)}_{\text{characteristic polynomial}}}.$$

Solution

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

Cases 1 & 3

$$Y(s) = \frac{2s^2+10s}{(s+1)\underbrace{(s^2-2s+5)}} = \frac{2s^2+10s}{(s+1)((s-1)^2+2^2)}$$

$D = 2^2 - 4\cdot 5 < 0$

Complete squares

$s^2 - 2s + 1 - 1 + 5 = (s-1)^2 + 2^2$

$$= \frac{A}{s+1} + \frac{B(s-1) + 2C}{(s-1)^2 + 2^2}$$

$$= \frac{A((s-1)^2+2^2) + (B(s-1)+2C)(s+1)}{(s+1)((s-1)^2+2^2)}$$

$$2s^2 + 10s = A((s-1)^2+2^2) + (B(s-1)+2C)(s+1)$$

$s = 1$: $\quad 2 + 10 = 4A + 4C \Rightarrow 12 = 4(A+C)$

$\Rightarrow A + C = 3 \Rightarrow C = 3 - A$

$s = -1$: $\quad 2 - 10 = (4+4)A \Rightarrow \boxed{A = -1}$

$\Rightarrow \boxed{C = 4}$

s^2: $\quad 2 = A + B \Rightarrow B = 2 - A \Rightarrow \boxed{B = 3}$

$$y(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{-\frac{1}{s+1} + \frac{3(s-1)+2\cdot 4}{(s-1)^2+2^2}\right\}$$

$$= -\underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}}_{\text{use Table}} + 3\,\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+2^2}\right\} + 4\,\mathcal{L}^{-1}\left\{\frac{2}{(s-1)^2+2^2}\right\}$$

$$\boxed{y(t) = -e^{-t} + 3e^{t}\cos 2t + 4e^{t}\sin 2t}$$

solution of the given IVP

9. EXAMPLE *Consider the IVP*

$$y'' + 4y' - 5y = te^t, \quad y(0) = 1, \quad y'(0) = 0. \tag{2}$$

SOLUTION (Main Steps): Application of Laplace Transform yields:

$$Y(s) = \frac{s^3 + 2s^2 - 7s + 5}{(s-1)^2(s^2 + 4s - 5)} = \frac{s^3 + 2s^2 - 7s + 5}{(s-1)^3(s+5)} \tag{3}$$

Partial Fraction Decomposition:

$$\frac{s^3 + 2s^2 - 7s + 5}{(s-1)^2(s^2 + 4s - 5)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^3} + \frac{D}{s+5}, \tag{4}$$

where

$$A = \frac{181}{216}, \quad B = -\frac{1}{36}, \quad C = \frac{1}{6}, \quad D = \frac{35}{216}.$$

Find the inverse Laplace Transform of $Y(S)$ (use Table (see Appendix 2)):

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{181}{216}e^t - \frac{1}{36}te^t + \frac{1}{12}t^2e^t + \frac{35}{216}e^{-5t}. \tag{5}$$

QUESTION: What is the general form of the solution of DE (2) by Method of Undetermined Coefficients?