20: Impulse Function (section 6.5)

1. In applications (mechanical systems, electrical circuits etc) one encounters functions (external force) of large magnitude that acts only for a very short period of time. To deal with violent forces of short duration the so called delta function is used. This function was introduces by Paul Dirac.

2. If a force $F(t)$ acts on a body of mass m on the time interval $[t_0, t_1]$, then the impulse due to F is defined by the integral

$$\text{impulse} = \int_{t_0}^{t_1} F(t)\mathrm{d}t = \int_{t_0}^{t_1} ma(t)\mathrm{d}t = \int_{t_0}^{t_1} m\frac{\mathrm{d}v(t)}{\mathrm{d}t}\mathrm{d}t = mv(t_1) - \underbrace{mv(t_0)}_{t=t_0}$$

momentum at $t=t_1$

3. The impulse equals the change in momentum.

When a hammer strikes an object, it transfers momentum to the object. This change in momentum takes place over a very short period of time. The change in momentum (=the impulse) is the area under the curve defined by $F(t)$

$$\text{the total impulse of the force } F(t) = \int_{-\infty}^{\infty} F(t)\mathrm{d}t$$

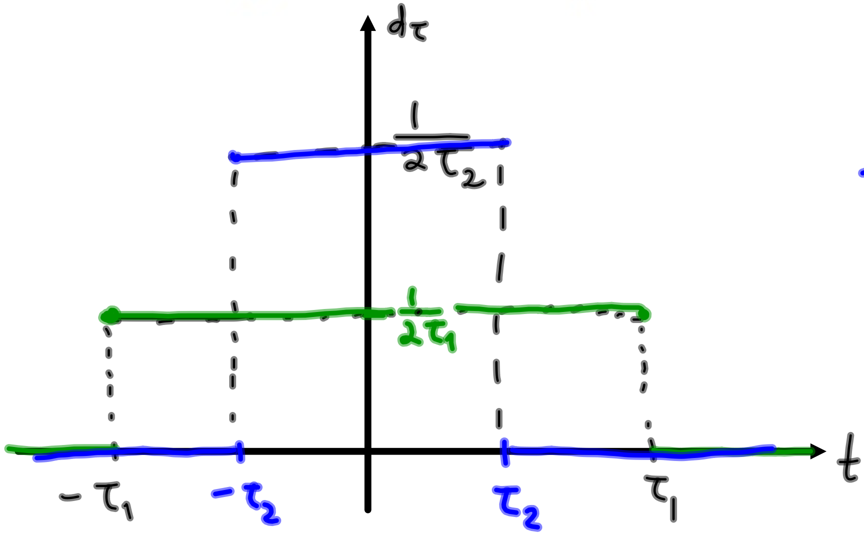
4. Consider a family of piecewise functions (forces)

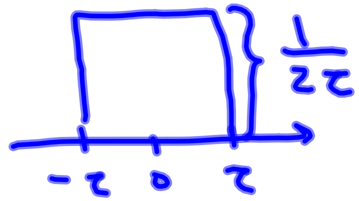
$$d_\tau = \begin{cases} \dfrac{1}{2\tau}, & \text{if } \ |t| < \tau \\ 0, & \text{if } \ |t| \geq \tau. \end{cases}$$

$-\tau < t < \tau$

$t \geq \tau, \ t \leq -\tau$

Then all forces d_τ have the total impulse which is equal 1.





$A = 2\tau \cdot \frac{1}{2\tau} = 1$

$\int_{-\infty}^{\infty} d_\tau \, dt = 1$

5. Dirac DELTA Function: In practice it is convenient to work with another type of unit impulse, an idealized unit impulse force that concentrated at $t = 0$:

$$\lim_{\tau \to 0} d_\tau(t) = \delta(t).$$

6. **Definition** The Dirac Delta Function, $\delta(t)$, is characterized by the following 2 properties:

   (a) $\delta(t)=0$ for all $t\neq 0$.

   (b) $\displaystyle\int_{-\infty}^{\infty}\delta(t)f(t)\mathrm{d}t=f(0)$ for any f$f(t)$ continuous on an open interval containing $t=0$.

   Note that δ-function does not behave like an ordinary function.

7. A unit impulse concentrated at $t=t_0$ is denoted by $\delta(t-t_0)$ and

$$\boxed{\int_{-\infty}^{\infty}\delta(t-t_0)f(t)\mathrm{d}t=f(t_0),\quad t\neq t_0.}$$

8. Laplace Transform of *delta*-function:

$$\boxed{\mathcal{L}\{\delta(t)\}=1}$$

For $t_0\geq 0$

$$\mathcal{L}\{\delta(t-t_0)\}=e^{-st_0}$$

$$\mathcal{L}\{\delta(t)\}=\int_0^{+\infty}\delta(t)e^{-st}dt=$$

$$=\int_{-\infty}^{+\infty}\delta(t)\underbrace{e^{-st}}_{f(t)}dt=f(0)=e^{-s\cdot 0}=e^0=1$$

11. Remark:

$$\int_{-\infty}^{t} \delta(t - t_0)\mathrm{d}t = \begin{cases} 0, & t < t_0 \\ 1, & t \geq t_0 \end{cases} = u_{t_0}(t).$$

In other words, derivative of unit step function is *delta*-function.

9. Solve the given IVP and sketch the graph of the solution:

$$y'' + y = \delta(t - 2\pi), \quad y(0) = 0, \quad y'(0) = 1.$$

$$\mathcal{L}\{y'' + y\} = \mathcal{L}\{\delta(t-2\pi)\}$$

$$\Downarrow$$

$$Y(s)(s^2+1) - 1 = e^{-2\pi s} \Rightarrow Y(s) = \left(e^{-2\pi s} + 1\right) \cdot \underbrace{\frac{1}{s^2+1}}_{G(s)}$$

$$G(s) \Downarrow g(t) = \sin t$$

$$Y(s) = e^{-2\pi s} G(s) + G(s)$$

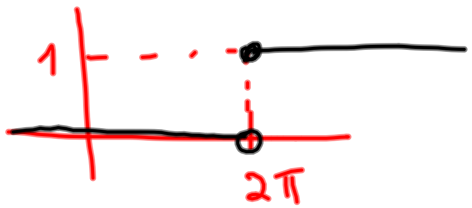
$$y(t) = \mathcal{L}^{-1}\{e^{-2\pi s} G(s)\} + \mathcal{L}^{-1}\{G(s)\}$$

$$y(t) = u_{2\pi}(t) g(t - 2\pi) + \sin t$$

$$y(t) = u_{2\pi}(t) \sin(t - 2\pi) + \sin t \Rightarrow y(t) = u_{2\pi}(t) \sin t + \sin t$$

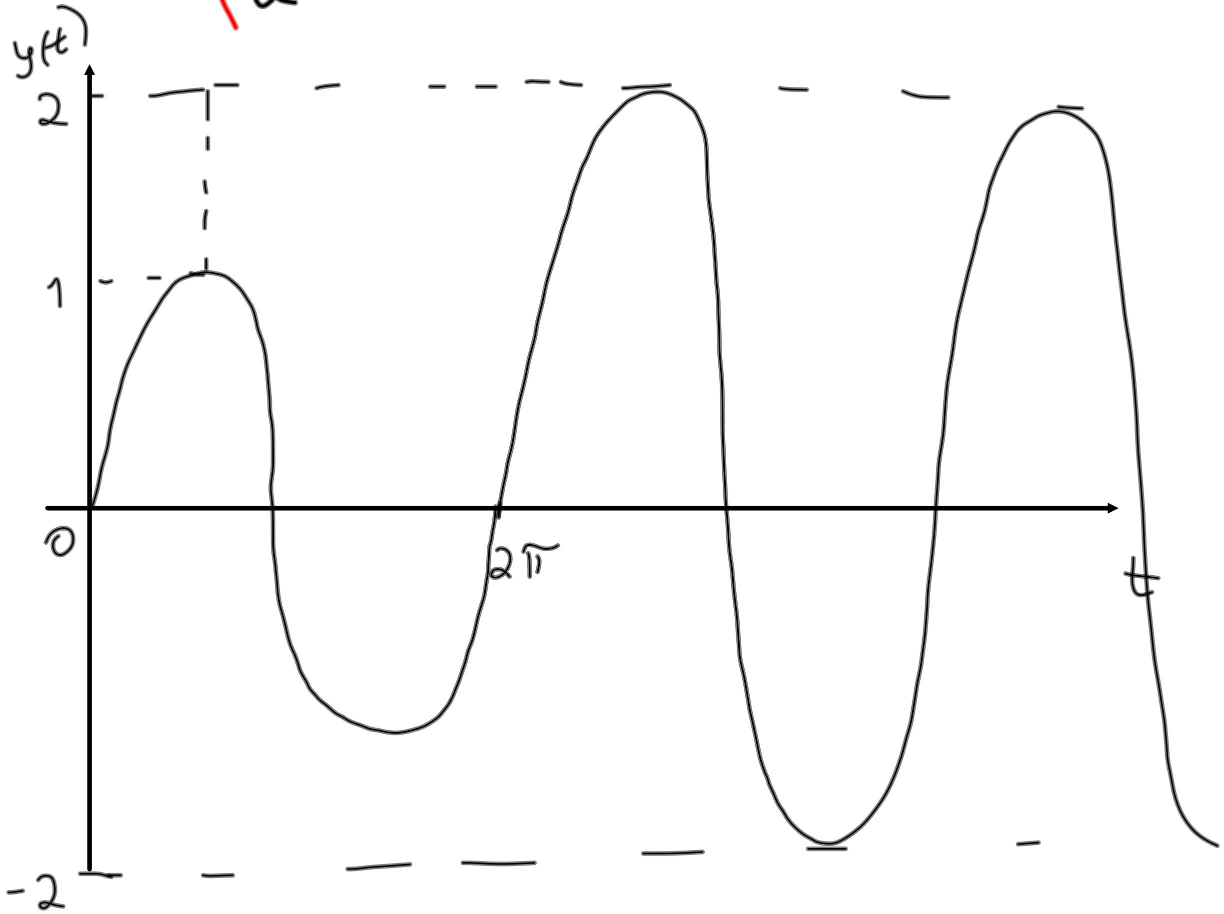
$$\longrightarrow$$

continue



Graph $y(t)$

$$y(t) = \begin{cases} \sin t & 0 \le t < 2\pi \\ 2\sin t & 2\pi \le t \end{cases}$$



10. Solve $2y'' + y' + 4y = \underbrace{\delta(t - \frac{\pi}{6})\sin t}$ subject to $y(0) = 0,\ y'(0) = 0.$

STEP 1

$$\mathcal{L}\{2y'' + y' + 4y\} = (2s^2 + s + 4)Y(s)$$

STEP 2

$$\mathcal{L}\{\delta(t - \tfrac{\pi}{6})\sin t\} = \int_0^{+\infty} \delta(t - \tfrac{\pi}{6})\sin t\, e^{-st}\,dt = \int_{-\infty}^{+\infty} \delta(t - \tfrac{\pi}{6})\underbrace{\sin t\, e^{-st}}_{f(t)}\,dt$$

$$= f\left(\tfrac{\pi}{6}\right) = \sin\tfrac{\pi}{6}\, e^{-\frac{\pi}{6}s} = \tfrac{1}{2}e^{-\frac{\pi}{6}s}$$

STEP 3

$$(2s^2 + s + 4)Y(s) = \tfrac{1}{2}e^{-\frac{\pi}{6}s}$$

$$Y(s) = e^{-\frac{\pi}{6}s}\underbrace{\frac{1}{2(2s^2 + s + 4)}}_{G(s)} = e^{-\frac{\pi}{6}s}G(s)$$

STEP 4

$$y(t) = \mathcal{L}^{-1}\{e^{-\frac{\pi}{6}s}G(s)\} = u_{\frac{\pi}{6}}(t)\, g(t - \tfrac{\pi}{6})$$

$$G(s) = \frac{1}{2(2s^2 + s + 4)} = \frac{1}{4}\cdot\frac{1}{s^2 + \frac{s}{2} + 2} = \frac{1}{4}\,\frac{1}{\underbrace{s^2 + 2s\cdot\frac{1}{4} + (\frac{1}{4})^2}_{(s + \frac{1}{4})^2}\underbrace{-(\frac{1}{4})^2 + 2}_{\frac{31}{4}}}$$

$$= \frac{1}{4}\cdot\frac{1}{(s + \frac{1}{4})^2 + (\frac{\sqrt{31}}{2})^2} = \frac{1}{\cancel{4}_2}\cdot\frac{\cancel{2}}{\sqrt{31}}\,\frac{\sqrt{31}/2}{(s + \frac{1}{4})^2 + (\frac{\sqrt{31}}{2})^2}$$

$$g(t) = \frac{1}{\sqrt{31}}\,\mathcal{L}^{-1}\left\{\frac{\sqrt{31}/2}{(s + \frac{1}{4})^2 + (\frac{\sqrt{31}}{2})^2}\right\} = \frac{1}{2\sqrt{31}}e^{-\frac{t}{4}}\sin\left(\frac{\sqrt{31}}{2}t\right)$$

Finally,

$$y(t) = u_{\frac{\pi}{6}}(t)\,\frac{1}{2\sqrt{31}}\,e^{-\frac{(t - \frac{\pi}{6})}{4}}\sin\left(\frac{\sqrt{31}}{2}(t - \tfrac{\pi}{6})\right)$$