23: Appendix: Matrices and Matrix Multiplication (section 7.2)

(continued)

Determinant

11. Determinant of a matrix is a function that takes a square matrix and converts it into a number.

12. Determinant of 2×2 and 3×3 matrices.

   - A **determinant of order 2** is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

   - A **determinant of order 3** is defined by

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1$$

   - If the determinant of a matrix is zero we call that matrix **singular** and if the determinant of a matrix isnt zero we call the matrix **nonsingular.**

13. **Matrix Inverse**. Let A be a square matrix of size n. A square matrix, A^{-1}, of size n, such that $AA^{-1} = I_n$ (or , equivalently, $A^{-1}A = I_n$) is called an **inverse matrix**.

14. A^{-1} in the case $n = 2$: If $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ then

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $\det A = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4-6 = -2 \neq 0 \Rightarrow$ A is non singular $\Rightarrow$ inverse A^{-1} exists:

$$A^{-1} = \frac{1}{-2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

15. *FACT*: A^{-1} exists if and only if $\det A \neq 0$.

Equivalently:

- If A is nonsingular then A^{-1} will exist.
- If A is singular then A^{-1} will NOT exist.

16. **Solving Systems of Equations with Inverses.**

Let $AX = B$ be a linear system of n equations in n unknowns and A^{-1} exists, then $X = A^{-1}B$ is the *unique* solution of the system.

$$\underbrace{A^{-1}A}_{I}x = A^{-1}B$$

$$Ix = A^{-1}B$$

$$x = A^{-1}B$$

24: Basic Theory of Systems of First Order Linear Equations (sec. 7.4)

System of homogeneous linear equations (continued)

3. Consider IVP

$$X' = P(t)X, \quad \underbrace{X(t_0) = B}_{} \quad \Rightarrow \begin{pmatrix} x_1(t_0) \\ x_2(t_0) \\ \vdots \\ x_n(t_0) \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \qquad (2)$$

By Superposition Principle, if the vector functions

$$X_1(t) = \begin{pmatrix} x_{11}(t) \\ \vdots \\ x_{n1}(t) \end{pmatrix}, \ldots, X_n(t) = \begin{pmatrix} x_{1n}(t) \\ \vdots \\ x_{nn}(t) \end{pmatrix}$$

are solutions of the homogeneous system (1), then the linear combination

$$X(t) = C_1X_1(t) + \ldots + C_nX_n(t)$$

is also a solution for any constants $C_1, \ldots, C_n$.

Question: How to determine the the constants $C_1, \ldots, C_n$ corresponding to the given IVP?

$$B = X(t_0) = C_1 X_1(t_0) + \ldots + C_n X_n(t_0)$$

$$= C_1 \begin{pmatrix} x_{11}(t_0) \\ \vdots \\ x_{n1}(t_0) \end{pmatrix} + \ldots + C_n \begin{pmatrix} x_{1n}(t_0) \\ \vdots \\ x_{nn}(t_0) \end{pmatrix} = \begin{pmatrix} C_1 x_{11}(t_0) + \ldots + C_n x_{1n}(t_0) \\ \cdots \\ C_1 x_{n1}(t_0) + \ldots + C_n x_{nn}(t_0) \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} x_{11}(t_0) & \ldots & x_{1n}(t_0) \\ \cdot & \cdot & \cdot \\ x_{n1}(t_0) & \ldots & x_{nn}(t_0) \end{pmatrix}}_{\Psi(t_0)} \begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix}$$

4. Consider the matrix, whose columns are vectors $X_1(t), \ldots, X_n(t)$:

$$\Psi(t) = \begin{pmatrix} x_{11}(t) & \ldots & x_{1n}(t) \\ \vdots & \vdots & \vdots \\ x_{n1}(t) & \ldots & x_{nn}(t) \end{pmatrix}.$$

Then $B = X(t_0) = C_1X_1(t_0) + \ldots + C_nX_n(t_0) = \Psi(t_0)\begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix}$, equivalently,

$$\Psi(t_0)\begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix} = B \quad \Rightarrow \quad \begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix} = \Psi^{-1}(t_0) B$$

We can find a solution for any initial condition given by vector column B if and only if $det\Psi(t_0) \neq 0$. In other words, if WRONSKIAN $\neq 0$

5. Note that this determinant is called the **Wronskian** of the solutions $X_1, \ldots, X_n$ and is denoted by

$$W[X_1, \ldots, X_n](t) = \det \Psi(t).$$

6. Note that by analogy with section 3.2, $det\Psi(t_0) \neq 0$ implies $det\Psi(t) \neq 0$ for any t.

7. If $det\Psi(t) \neq 0$ then $X_1, \ldots, X_n$ is called the **fundamental set of solutions** and the general solution of the system (1) is $C_1X_1(t) + \ldots + C_nX_n(t)$.

9. Question: *How to find a fundamental set of solutions?* In the next section we answer it for the case $P(t) = const$, i.e. for system of linear homogeneous equations with constant coefficients.

$P(t) = A$"

10. Find general solution of $X' = AX$, where $A = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$.

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$X' = AX \iff \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 2x_1 \\ -x_2 \end{pmatrix} \Rightarrow \begin{cases} x_1' = 2x_1 \\ x_2' = -x_2 \end{cases}$$ uncoupled

$$x_1' = 2x_1 \Rightarrow x_1 = C_1 e^{2t}$$

$$x_2' = -x_2 \Rightarrow x_2 = C_2 e^{-t}$$

$\Rightarrow$ general solution

$$\boxed{X = \begin{pmatrix} c_1 e^{2t} \\ c_2 e^{-t} \end{pmatrix}}$$

Rewrite the general solution in alternate form

$$X = \begin{pmatrix} c_1 e^{2t} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ c_2 e^{-t} \end{pmatrix} = C_1 e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{-t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

In other words, A stretches twice in the direction of the x_1-axis.

$$A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$