25: Systems of Linear Algebraic Equations. Eigenvalues and Eigenvectors (section 7.3)

Eigenvalues and Eigenvectors

1. A number λ is called an **eigenvalue** of matrix A if there exists a **nonzero** vector v such that

$$Av = \lambda v,$$

and v is called an **eigenvector** corresponding to the eigenvalue λ.

2. Example. If A is diagonal matrix,

$$A = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

then the numbers $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigenvalues and the vectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \quad v_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix},$$

are the corresponding eigenvectors.

3. How to find eigenvalues?

Eigenvalue are solutions of the following **characteristic equation (polynomial):**

$$\det(A - \lambda I) = 0.$$

4. Show that the characteristic equation in the case $n = 2$ can be found as

$$\lambda^2 - \mathrm{trace}(A)\lambda + \det(A) = 0.$$

6. *Example.* Find eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} -2 & 1 \\ 2 & -3 \end{pmatrix}$.

5. *Remark.* For $n \times n$ matrix the characteristic equation is a polynomial equation of degree n. The eigenvectors corresponding to λ can be found by solving the corresponding system of linear equations $(A - \lambda I)v = 0$ (as we will see in the next examples).

7. *Example.* Given

$$A = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$$

(a) Find eigenvalues of A.

$$\det(A-\lambda I) = -(\lambda-1)(\lambda+2)(\lambda-3) = 0$$

$$\lambda_1 = 1, \quad \lambda_2 = -2, \quad \lambda_3 = 3$$

(b) Find eigenvectors of A (use Gauss Elimination Method below).

$$(A-\lambda I)v = 0$$

$\boxed{\lambda = 1}$ $\quad (A - I)v = 0 \Rightarrow \begin{pmatrix} 0 & -1 & 4 \\ 3 & 1 & -1 \\ 2 & 1 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Augmented matrix

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left(\begin{array}{ccc|c} 0 & -1 & 4 & 0 \\ 3 & 1 & -1 & 0 \\ 2 & 1 & -2 & 0 \end{array}\right) \xrightarrow{R_3 \leftrightarrow R_1} \left(\begin{array}{ccc|c} 2 & 1 & -2 & 0 \\ 3 & 1 & -1 & 0 \\ 0 & -1 & 4 & 0 \end{array}\right)$$

$$\xrightarrow{R_2 \leftrightarrow R_2 + \left(-\frac{3}{2}\right)R_1} \left(\begin{array}{ccc|c} 2 & 1 & -2 & 0 \\ 0 & -\frac{1}{2} & 2 & 0 \\ 0 & -1 & 4 & 0 \end{array}\right) \xrightarrow{R_3 \leftrightarrow R_3 - 2R_2}$$

$$\left(\begin{array}{ccc|c} 2 & 1 & -2 & 0 \\ 0 & -\frac{1}{2} & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$2v_1 + v_2 - 2v_3 = 0$ $\qquad v_3 = 1$

$-\frac{1}{2}v_2 + 2v_3 = 0 \Rightarrow -\frac{1}{2}v_2 + 2 = 0 \Rightarrow v_2 = 4$

$\longrightarrow 2v_1 + 4 - 2 = 0 \Rightarrow v_1 = -1$

An Eigenvector corresponding to $\lambda_1 = 1$ is

$$v^1 = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$$

$\boxed{\lambda = -2}$ $(A+2I)v=0 \Rightarrow \begin{pmatrix} 3 & -1 & 4 \\ 3 & 4 & -1 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Augmented matrix

$$\left(\begin{array}{ccc|c} 3 & -1 & 4 & 0 \\ 3 & 4 & -1 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right) \xrightarrow{R_3 \leftrightarrow R_3 - \frac{2}{3}R_2} \left(\begin{array}{ccc|c} 3 & -1 & 4 & 0 \\ 3 & 4 & -1 & 0 \\ 0 & -\frac{5}{3} & \frac{5}{3} & 0 \end{array}\right)$$

$$\xrightarrow[R_3 \leftrightarrow 3R_3]{R_2 \leftrightarrow R_2 - R_1} \left(\begin{array}{ccc|c} 3 & -1 & 4 & 0 \\ 0 & 5 & -5 & 0 \\ 0 & -5 & 5 & 0 \end{array}\right) \xrightarrow{R_3 \leftrightarrow R_3 + R_2} \left(\begin{array}{ccc|c} 3 & -1 & 4 & 0 \\ 0 & +5 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$3v_1 - v_2 + 4v_3 = 0$$
$$5v_2 - 5v_3 = 0$$

$v_3 = 1$

$v_2 = v_3 = 1$

$v_1 = \frac{1}{3}(v_2 - 4v_3) = \frac{1}{3}(1-4) = -1$

For λ_2 we have

$$v_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$\boxed{\lambda=3}$ $\quad (A-3I)v=0 \Rightarrow \begin{pmatrix} -2 & -1 & 4 \\ 3 & -1 & -1 \\ 2 & 1 & -4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Augmented matrix

$$\left(\begin{array}{ccc|c} -2 & -1 & 4 & 0 \\ 3 & -1 & -1 & 0 \\ 2 & 1 & -4 & 0 \end{array}\right) \xrightarrow{R_3 \leftrightarrow R_3+R_1} \left(\begin{array}{ccc|c} -2 & -1 & 4 & 0 \\ 3 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_2 \leftrightarrow R_2+\frac{3}{2}R_1} \left(\begin{array}{ccc|c} -2 & -1 & 4 & 0 \\ 0 & -\frac{5}{2} & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} -2v_1 - v_2 + 4v_3 = 0 \\ -\frac{5}{2}v_1 + 5v_2 = 0 \end{cases}$$

$v_2 = 1 \Rightarrow -\frac{5}{2}v_1 + 5 = 0 \Rightarrow v_1 = 2$

$v_3 = \frac{1}{4}(2v_1 + v_2) = \frac{1}{4}(2+2) = 1$

$$v^3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$