27: Complex Eigenvalues (section 7.6)

1. When the characteristic equation has real coefficients, complex eigenvalues always appear in conjugate pairs.

If λ is an eigenvalue of $A \Rightarrow \bar{\lambda}$ is also its eigenvalue

If A is real $\Rightarrow \overline{Av} = \overline{\lambda v} \Rightarrow A\bar{v} = \bar{\lambda}\bar{v} \Rightarrow \bar{v}$ is an eigenvector corresponding to $\bar{\lambda}$

$A = \bar{A}$

2. Case $n = 2$. If $\lambda = \alpha + i\beta$ is a complex eigenvalue of the coefficient matrix A in the homogeneous system $X' = AX$ and v is a corresponding eigenvector then

   - $$\left\{e^{\lambda t}v, e^{\bar{\lambda}t}\bar{v}\right\}$$
     is a fundamental set of solutions of the system $X' = AX$.

     For example, If A is 2×2 s.t. $\lambda_1 = 2+3i$ and $v_1 = \begin{pmatrix} 1+i \\ 3 \end{pmatrix}$

     Then Fundam. set $\left\{ e^{(2+3i)t}\begin{pmatrix} 1+i \\ 3 \end{pmatrix}, e^{(2-3i)t}\begin{pmatrix} 1-i \\ 3 \end{pmatrix} \right\}$

   - $$\left\{\mathrm{Re}(e^{\lambda t}v), \mathrm{Im}(e^{\lambda t}v)\right\}$$
     is a **real** fundamental set of solutions of the system $X' = AX$.

3. Example. *Consider* $\begin{pmatrix} 3 & 1 \\ -5 & 1 \end{pmatrix}$

(a) *Find general solution of the system* $X' = AX$.

$\operatorname{tr} A = 3+1 = 4$

$\det A = 3-(-5) = 8$

Find eigenvalues $\det(A - \lambda I) = 0$

$\lambda^2 - (\operatorname{tr} A)\lambda + (\det A) = 0$

$\lambda^2 - 4\lambda + 8 = 0$, $D = 4^2 - 4\cdot 8 < 0$

$(\lambda-2)^2 + 4 = 0 \Rightarrow \lambda = 2 \pm 2i$

Find an eigenvector v corresponding to $\lambda = 2+2i$

$(A - \lambda I)v = 0 \Rightarrow \left(\begin{array}{cc|c} 3-(2+2i) & 1 & 0 \\ -5 & 1-(2+2i) & 0 \end{array}\right)$

$\left(\begin{array}{cc|c} 1-2i & 1 & 0 \\ -5 & -(1+2i) & 0 \end{array}\right) \xrightarrow{R_2 \leftrightarrow \frac{(1-2i)}{5} R_2} \left(\begin{array}{cc|c} 1-2i & 1 & 0 \\ -(1-2i) & -1 & 0 \end{array}\right) \rightarrow$

$-(1+2i)\dfrac{(1-2i)}{5} = -\frac{1}{5}(1^2+2^2) = -1$

$\xrightarrow{R_2 \leftrightarrow R_2 + R_1} \left(\begin{array}{cc|c} 1-2i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \rightarrow$ $(1-2i)v_1 + v_2 = 0$

Set $v_1 = 1 \Rightarrow v_2 = -1+2i$

Conclusion $\lambda = 2+2i$, $v = \begin{pmatrix} 1 \\ -1+2i \end{pmatrix}$

Fundam. set

$$\left\{ e^{(2+2i)t}\begin{pmatrix} 1 \\ -1+2i \end{pmatrix}, \; e^{(2-2i)t}\begin{pmatrix} 1 \\ -1-2i \end{pmatrix} \right\}$$

General (complex valued) solution

$$X = C_1 e^{(2+2i)t}\begin{pmatrix} 1 \\ -1+2i \end{pmatrix} + C_2 e^{(2-2i)t}\begin{pmatrix} 1 \\ -1-2i \end{pmatrix}$$

(b) Find general real solution of the system $X' = AX$.

(c) Find solution subject to the initial conditions $x_1(0) = 2, x_2(0) = 3$.

4. If $\lambda = \alpha + i\beta$ is a complex eigenvalue of a real matrix A and $v = a + ib$ is a corresponding eigenvector, then

$$\mathrm{Re}(e^{\lambda t} v) = e^{\alpha t}(a\cos(\beta t) - b\sin(\beta t)), \quad \mathrm{Im}(e^{\lambda t} v) = e^{\alpha t}(a\sin(\beta t) + b\cos(\beta t)).$$

5. Case $n = 3$. If $\alpha \pm i\beta$ are complex eigenvalue of the coefficient matrix A, then the third eigenvalue must be real (denote it by λ). Let v and w be eigenvectors corresponding to $\alpha + i\beta$ and λ, respectively. Then

$$\left\{\mathrm{Re}(e^{\lambda t}v), \mathrm{Im}(e^{\lambda t}v), e^{\lambda t}w\right\}$$

is a **real** fundamental set of solutions of the system $X' = AX$.