28: Repeated Eigenvalues (sec. 7.5 (continued) and sec. 7.8)

10. Example. *Consider the system:*

$$\begin{aligned} x_1' &= -3x_1 + \tfrac{5}{2}x_2 \\ x_2' &= -\tfrac{5}{2}x_1 + 2x_2 \end{aligned}$$

(a) *Find general solution of the system.*

(b) *Find solution of the system satisfying* $x_1(0) = 2, x_2(0) = 1$.

Previously we found $\lambda = -\frac{1}{2}$, $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $w = \begin{pmatrix} 0 \\ 2/5 \end{pmatrix}$

Fundamental set $\left\{ e^{-\frac{1}{2}t}\begin{pmatrix} 1 \\ 1 \end{pmatrix},\ te^{-\frac{1}{2}t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-\frac{1}{2}t}\begin{pmatrix} 0 \\ 2/5 \end{pmatrix} \right\}$

$$\Psi(t) = \begin{pmatrix} e^{-\frac{1}{2}t} & te^{-\frac{1}{2}t} \\ e^{-\frac{1}{2}t} & te^{-\frac{1}{2}t} + \frac{2}{5}e^{-\frac{1}{2}t} \end{pmatrix}$$

fundamental matrix

General solution

$$\boxed{X(t) = C_1\begin{pmatrix} e^{-\frac{1}{2}t} \\ e^{-\frac{1}{2}t} \end{pmatrix} + C_2\begin{pmatrix} te^{-\frac{1}{2}t} \\ te^{-\frac{1}{2}t} + \frac{2}{5}e^{-\frac{1}{2}t} \end{pmatrix}}$$

Note

$$X(t) = \begin{pmatrix} e^{-\frac{1}{2}t}C_1 + te^{-\frac{1}{2}t}C_2 \\ e^{-\frac{1}{2}t}C_1 + \left(te^{-\frac{1}{2}t} + \frac{2}{5}e^{-\frac{1}{2}t}\right)C_2 \end{pmatrix} = \Psi(t)\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \Psi(t)C$$

$$\boxed{X(t) = \Psi(t)C}$$

(b) $X(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$X(t) = \Psi(t) C \Rightarrow X(0) = \Psi(0) C$

$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = C = \Psi^{-1}(0) X(0) = \Psi^{-1}(0) \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\Psi(0) = \begin{pmatrix} 1 & 0 \\ 1 & 2/5 \end{pmatrix} \Rightarrow \Psi^{-1}(0) = \frac{1}{\frac{2}{5}} \begin{pmatrix} 2/5 & 0 \\ -1 & 1 \end{pmatrix}$

$\det \Psi(0) = \frac{2}{5}$

$\Psi^{-1}(0) = \begin{pmatrix} 1 & 0 \\ -\frac{5}{2} & \frac{5}{2} \end{pmatrix}$

$C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -5/2 & 5/2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -\frac{5}{2} \end{pmatrix}$

$c_1 = 2 \qquad c_2 = -\frac{5}{2}$

$$x(t) = 2 \begin{pmatrix} e^{-t/2} \\ e^{-t/2} \end{pmatrix} - \frac{5}{2} \begin{pmatrix} t e^{-t/2} \\ t e^{-t/2} + \frac{2}{5} e^{-t/2} \end{pmatrix}$$