# 30: Non-homogeneous Linear Systems (section 7.9)

1. Consider a Non-homogeneous Linear system

$$X' = P(t)X + G(t), \tag{1}$$

where

$$X = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad P(t) = \begin{pmatrix} p_{11}(t) & p_{12}(t) & \dots & p_{1n}(t) \\ p_{21}(t) & p_{22}(t) & \dots & p_{2n}(t) \\ \vdots & & & \vdots \\ p_{n1}(t) & p_{n2}(t) & \dots & p_{nn}(t) \end{pmatrix}, \quad G(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{pmatrix},$$

2. By analogy with the case of a scalar equation we have here that general solution can be expressed as

$$X(t) = X_p(t) + X_H(t),$$

where $X_p$ is a particular solution of the given system, and $X_h$ is a solution of the corresponding homogeneous system,

$$X' = P(t)X. \tag{2}$$

3. Suppose that $\{X_1(t), \ldots, X_n(t)\}$ is a fundamental set solutions of the corresponding homogeneous system (2). Consider the so called **fundamental matrix**, $\Psi(t)$, whose columns are vectors $X_1(t), \ldots, X_n(t)$ [1]

$$\begin{array}{c} X_1 \\ \end{array}$$

$$\Psi(t) = \begin{pmatrix} x_{11}(t) & \ldots & x_{1n}(t) \\ \vdots & \vdots & \vdots \\ x_{n1}(t) & \ldots & x_{nn}(t) \end{pmatrix}.$$

Then

$$X_h(t) = \Psi(t)C, \tag{3}$$

variate C

where $C = \begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix}$.

Note that if $\{X_1, \ldots, X_n\}$ fundam. set of solutions for $x' = Px$ then $\boxed{\Psi' = P\Psi}$ ☆

**Method of Variation of Parameters**

4. We use the above method to find $X_p(t)$. Seek a particular solution of (1) variating parameters:

$$X(t) = \Psi(t)U(t) = x_p(t)$$

where $U(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{pmatrix}$.

To determine $U(t)$ we substitute this part. solution $x = \Psi U$ into the given non-homogeneous system

$$x' = Px + G$$

$$x' = (\Psi U)' = \Psi' U + \Psi U' \stackrel{\star}{=} P\Psi U + \Psi U'$$

$$x' = Px + G = P\Psi U + G$$

$$\Psi U' = G$$

$$\Psi^{-1}\Psi U' = \Psi^{-1} G$$

$$\boxed{U' = \Psi^{-1} G}$$

5. One can show that then

$$\Psi(t)U'(t) = G(t),$$

which implies

$$U'(t) = \Psi^{-1}(t)\, G(t)$$

$$\Rightarrow \quad U(t) = \int_0^t \Psi^{-1}(\tau) G(\tau)\, d\tau$$

$$x_p = \Psi(t)\, U(t)$$

$$X_p(t) = \int_0^t \Psi(t)\Psi^{-1}(\tau)g(\tau)\mathrm{d}\tau.$$

As a result we get general solution of (1):

$$X(t) = \underbrace{\Psi(t)C}_{\text{General Solution of } x' = Px} + \underbrace{\int_0^t \Psi(t)\Psi^{-1}(\tau)g(\tau)\mathrm{d}\tau}_{X_p}.$$

6. Example Find general solution of the system: $X(t) = X_h(t) + X_p(t)$

$$x_1' = -2x_1 + x_2 + e^{-t}$$
$$x_2' = x_1 - 2x_2 - e^{-t}$$

$\Rightarrow A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$

$G(t) = \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix}$

where $0 < t < \pi$.

Find solution of $X' = AX$

One can get $\lambda_1 = -3$, $\lambda_2 = -1$ Check it

$v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Fundamental set $\left\{ e^{-3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$

$$X_h(t) = c_1 e^{-3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \Psi(t) C = \Psi(t)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Fundam. matrix $\Psi(t) = \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix}$

Variate parameters to find $X_p(t)$:

$$X(t) = \Psi(t)\, U(t), \quad \text{where } U(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$$

We know that $U'(t) = \Psi^{-1}(t)\, G(t)$

Find $\Psi^{-1}(t)$ !

$$\text{Wronskian} = \det \Psi(t) = e^{-4t} + e^{-4t} = 2e^{-4t}$$

$$\Psi^{-1}(t) = \frac{1}{2e^{-4t}} \begin{pmatrix} e^{-t} & -e^{-t} \\ e^{-3t} & e^{-3t} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} e^{3t} & -\frac{1}{2} e^{3t} \\ \frac{1}{2} e^{t} & \frac{1}{2} e^{t} \end{pmatrix}$$

$$u'(t) = \Psi^{-1}(t) G(t)$$

$$= \begin{pmatrix} \frac{1}{2} e^{3t} & -\frac{1}{2} e^{3t} \\ \frac{1}{2} e^{t} & \frac{1}{2} e^{t} \end{pmatrix} \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix}$$

$$= \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix} \Rightarrow u(t) = \begin{pmatrix} \frac{1}{2} e^{2t} \\ 0 \end{pmatrix}$$

$$x_p(t) = \psi(t)\, u(t)$$

$$= \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix} \begin{pmatrix} \frac{1}{2} e^{2t} \\ 0 \end{pmatrix}$$

$$x_p(t) = \begin{pmatrix} \frac{1}{2} e^{-t} \\ -\frac{1}{2} e^{-t} \end{pmatrix}$$

Finally, general solution

$$x(t) = \underbrace{c_1 e^{-3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{x_h} + \underbrace{\begin{pmatrix} \frac{1}{2} e^{-t} \\ -\frac{1}{2} e^{-t} \end{pmatrix}}_{x_p}$$