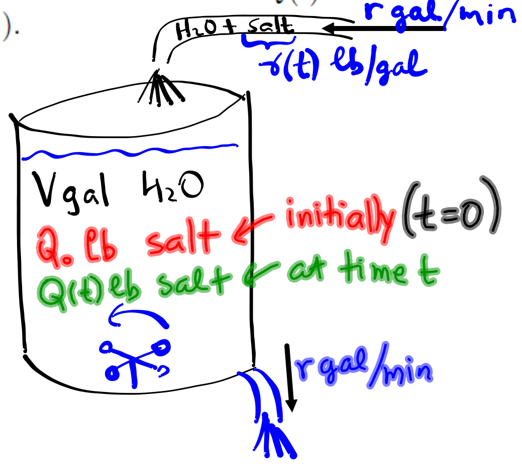
4. Modeling with First Order ODE (section 2.3)

1. An ODE that describes some physical process is called **mathematical model.**

2. Reminder: **A model of free-fall motion retarded by air resistance** was considered in Section 1 (#13) of these notes.

3. **Mixing Problems** serve as a model for problem of discharge and filtration of pollutants in a river, injection and absorption of medication in the bloodstream, for example.

4. *Chemicals in Tank, or Mixing.* Assume that a tank contains V gal of water and Q_0 lb of some substance (salt, or chemical, for example) is dissolved in it. Suppose that the water containing $\gamma(t)$ lb/gal of substance is entering the tank with the rate r gal/min (the concentration of the chemicals in the entering water varies in time) and then well stirred mixture is draining from the tank at the same rate. Find the amount $Q(t)$ of the substance in tank at any time (Do not solve, just find IVP for $Q(t)$).



In this problem
$V, \gamma(t), Q_0$ and r are given
$Q(t)$ is unknown function

$\dfrac{dQ}{dt}$ rate of change of amount of salt at tank

$$\frac{dQ}{dt} = \begin{pmatrix}\text{rate in}\\ \text{of salt}\end{pmatrix} - \begin{pmatrix}\text{rate out}\\ \text{of salt}\end{pmatrix}$$

$$\begin{pmatrix}\text{rate in}\\ \text{of salt}\end{pmatrix} = \begin{pmatrix}\text{salt concentration}\\ \text{before tank entering}\end{pmatrix} \cdot \begin{pmatrix}\text{rate in}\\ \text{of solution}\end{pmatrix} = \gamma(t)\cdot r$$

$$\begin{pmatrix}\text{rate out}\\ \text{of salt}\end{pmatrix} = \begin{pmatrix}\text{salt concentration}\\ \text{in tank after}\\ \text{mixing}\end{pmatrix} \begin{pmatrix}\text{rate out}\\ \text{of solution}\end{pmatrix} = \frac{Q(t)}{V}\cdot r$$

$$\boxed{\frac{dQ}{dt} = \gamma(t) r - \frac{Q(t)}{V}\cdot r}$$

Rewrite it

$$\text{IVP}\begin{cases} Q' + \dfrac{r}{V} Q = \gamma(t) r \\ Q(0) = Q_0 \end{cases}$$

1-st order Linear ODE

We can use Method of integrating factor to find the solution.

5. *Chemicals in Pond* (from the textbook) A pond contains 10 million gal of fresh water. Stream water containing an undesirable chemical flows into the pond at the rate 5 million gal/yr, and the mixture in the pond flows out through an overflow culvert at the same rate. The concentration $\gamma(t)$ of chemical in the incoming water varies periodically with time t, measured in years, according to the expression $\gamma(t) = 2 + \sin(2t)$g/gal.

(a) Construct a mathematical model of this flow process.

$V = 10 \cdot 10^6 \text{ gal} = 10^7 \text{ gal}$

$r = 5 \cdot 10^6 \text{ gal/yr}$

$\gamma(t) = 2 + \sin(2t) \text{ g/gal}$

$Q(t)$ amount of chemicals in pond at time t (g/gal)

$$\begin{cases} Q' + \dfrac{5 \cdot 10^6}{10^7} Q = (2 + \sin 2t) \cdot 5 \cdot 10^6 \\ Q(0) = 0 \end{cases}$$

$$\begin{cases} Q' + \frac{1}{2} Q = (10 + 5 \sin 2t) \cdot 10^6 \\ Q(0) = 0 \end{cases}$$

(b) Determine the amount of chemical in the pond at any time.

Set $Q(t) = 10^6 q(t)$

Put it into $(*)$

$$\begin{cases} \cancel{10^6} q' + \frac{1}{2}\cdot \cancel{10^6} q = (2+\sin 2t)\cdot 5\cdot \cancel{10^6} \\ \cancel{10^6} q(0) = 0 \end{cases}$$

$$\boxed{q' + \frac{1}{2} q = 10 + 5\sin 2t, \quad q(0) = 0}$$

Apply Method of Integrating Factor to solve

Multiply by $\mu(t)$:

$$\mu q' + \underbrace{\frac{\mu}{2}}_{=\mu'} q = 10\mu + 5\mu\sin(2t)$$

Find μ:

$$\mu' = \frac{\mu}{2}$$

$$\mu = e^{\frac{t}{2}}$$

$$(\mu q)' = 10\mu + 5\mu\sin(2t)$$

$$\mu q = 10\int \mu\, dt + 5\int \mu \sin 2t\, dt$$

$$e^{\frac{t}{2}} q = 10\int e^{\frac{t}{2}} dt + 5\int e^{\frac{t}{2}} \sin 2t\, dt$$

$$e^{\frac{t}{2}} q = 10\cdot 2 e^{t/2} + 5\frac{e^{t/2}}{\frac{1}{4}+4}\left(\frac{1}{2}\sin 2t - 2\cos 2t\right) + C$$

Reminder from Math 152

$$\int e^{at}\sin bt\, dt = \frac{e^{at}}{a^2+b^2}(a\sin bt - b\cos bt)$$

In our case $a = \frac{1}{2}$ $b = 2$

$$e^{\frac{t}{2}} q = 20 e^{\frac{t}{2}} + \frac{20}{17} e^{t/2}\left(\frac{1}{2}\sin 2t - 2\cos 2t\right) + C$$

$$\underbrace{q(t) = 20 + \frac{10}{17}\sin 2t - \frac{40}{17}\cos 2t + C e^{-t/2}}_{\text{general solution of}}$$

Find solution of IVP:

$q(0) = 0$

$$0 = q(0) = 20 + 0 - \frac{40}{17} + C \Rightarrow C = \frac{40}{17} - 20 = -\frac{300}{17}$$

$$q(t) = 20 + \frac{10}{17}\sin 2t - \frac{40}{17}\cos 2t - \frac{300}{17}e^{-t/2}$$

Solution of the original IVP is

$$\boxed{Q(t) = 10^6\left(\underbrace{20 + \frac{10}{17}\sin 2t - \frac{40}{17}\cos 2t}_{\text{steady-state solution}} \underbrace{- \frac{300}{17}e^{-t/2}}_{\text{transient solution}}\right)}$$

$$- \frac{300}{17}e^{-t/2} \xrightarrow{t \to +\infty} 0$$

(c) Plot the solution and describe in words the effect of the variation of the incoming concentration.

$\underline{\qquad\qquad}$ $\gamma(t)$

$\gamma(t) = 2 + 2\sin 2t$