7. Differences between Linear and Nonlinear Equations (section 2.4)

1. **Existence and Uniqueness of Solutions**

   THEOREM 1. *Let the functions f and $\partial f/\partial y$ be continuous in some rectangle*

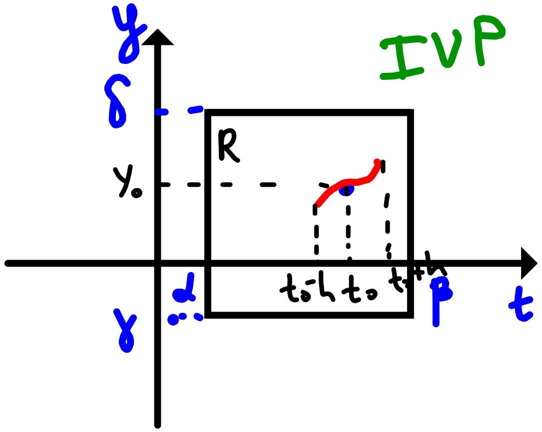
   $$R = \{(t, y) | \alpha < t < \beta, \quad \gamma < y < \delta\}$$

   containing the point (t_0, y_0). Then in some interval $t_0 - h < t < t_0 + h$ contained in $I = \{t|\ \alpha < t < \beta\}$, there is a **unique** *solution $y = y(t)$ of the initial value problem*

   IVP

   $$\boxed{y' = f(t, y), \quad y(t_0) = y_0.}$$

   1st order DE



① $f(t,y)$ is continuous in R

② $\dfrac{\partial f}{\partial y}$ is cont. in R

2. By this theorem we can guarantee the existence of solution only for values of t which are sufficiently closed to t_0, but not for all t.

3. Geometric consequence of the theorem is that two integral curves never intersect each other.

4. The condition "$\partial f/\partial y$ be continuous in some rectangle..." is important for uniqueness.
Illustration: Apply the existence and uniqueness theorem to the following IVP:

$$y' = y^{2/3}, \quad y(0) = 0.$$

$f(t,y) = y^{2/3}$ continuous for all (t,y)

$$\frac{\partial f}{\partial y} = \frac{2}{3} y^{-1/3} = \frac{2}{3y^{1/3}}$$

continuous for all (t,y) such that $y \neq 0$

We can not apply Theorem 1 here

Let us solve it: $y' = y^{2/3}$, $y(0)=0$

$\frac{dy}{dt} = y^{2/3}$, $y \neq 0$

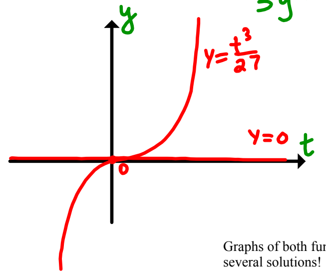
$\boxed{y=0}$ is a solution of the given IVP

$\int y^{-\frac{2}{3}} dy = \int dt$

$3y^{1/3} = t + C$

Find C: $y(0)=0 \Rightarrow 3 \cdot 0 = 0 + C \Rightarrow C=0$

We have one more solution of IVP: $3y^{1/3} = t$, or $\boxed{y = \frac{t^3}{27}}$



no uniqueness for this IVP

Graphs of both functions pass through the same point (0,0). In other words an IVP can have several solutions!

5. **Existence and Uniqueness of Solutions of <u>Linear</u> ODE**

THEOREM 2. *If the functions $p(t)$ and $g(t)$ are continuous on the interval $I = \{t \mid \alpha < t < \beta\}$, then for ant $t = t_0$ on I, there is a* **unique** *solution $y = y(t)$ of the initial value problem*

$$\boxed{y' + p(t)y = g(t), \quad y(t_0) = y_0.} \tag{1}$$

6. Note that the conditions of the Theorm 1 hold automatically for linear ODE.

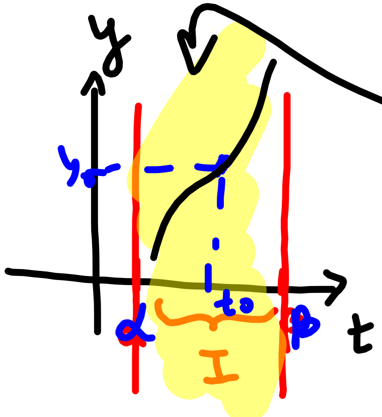
$$y' = \underbrace{g(t) - p(t)y}_{f(t,y)}$$

If $p(t)$ & $g(t)$ are cont. on I

then

① $f(t,y)$ is cont. on $\{(t,y) \mid \alpha \le t \le \beta, -\infty < y < \infty\}$

② $\dfrac{\partial f}{\partial y} = \dfrac{\partial}{\partial y}\left(g(t) - p(t)y\right) = -p(t)$ cont.



7. Determine (without solving the problem) an interval in which the solution of the given IVP is certain to exist:

$$(t-3)y' + (\ln|t|)y = 2t$$

linear 1st order ODE

(a) $y(1) = 2$

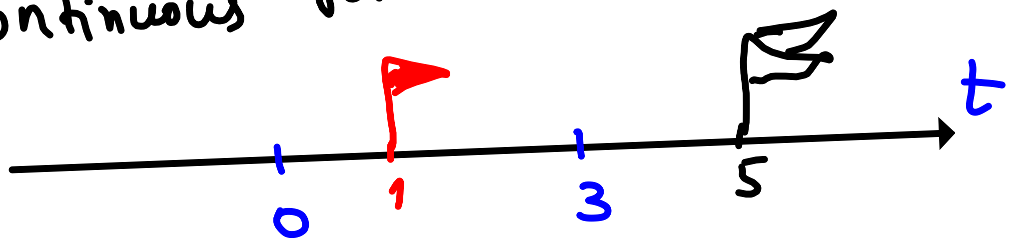
(b) $y(5) = 7$

$$y' + \underbrace{\frac{\ln|t|}{t-3}}_{p(t)}y = \underbrace{\frac{2t}{t-3}}_{g(t)}$$

Apply Theorem 2 :

$p(t)$ is continuous for all $t \neq 3, 0$

$g(t)$ is continuous for all $t \neq 3$



(a) $y(1)=2$ $\boxed{(0,3)}$

(b) $y(5)=7$ $\boxed{(3,+\infty)}$

8. Consider

$$ty' + 2y = 4t^2 \qquad \text{Linear} \tag{2}$$

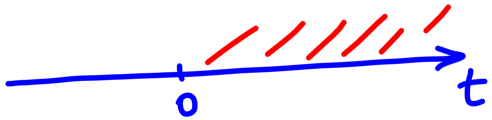
(a) Determine (without solving the problem) an interval in which the solution (2) satisfying $y(t_0) = y_0$ with $t_0 > 0$ is certain to exist.

IVP $\quad ty' + 2y = 4t^2$ $\xrightarrow{\text{Apply Th.2}}$ $y' + \underbrace{\frac{2}{t}}_{p(t)} y = \underbrace{4t}_{g(t)}$

$\quad y(t_0) = y_0 \quad (t_0 > 0)$

p & g continuous for all $t \neq 0$

$\boxed{(0, +\infty)}$



(b) The solution of IVP from item (a) found by the method of integrating factor is given:

$$y(t) = t^2 + \frac{C}{t^2}, \qquad C = t_0^2(y_0 - t_0^2). \qquad \text{(Check it!!!)}$$

Using that information discuss the domain of the solution and compare your conclusion with the answer of (a).