EXAMPLE 8. *Given differential equation*

$$xy'' + y' + xy = 0.$$

a) Seek power series solutions of this equation about $x_0 = 1$: find the recurrence relation for coefficients of the power series about $x_0 = 1$ representing a solution (in general, a recurrence relation is a relation expressing the nth coefficients a_n in terms of some previous ones).

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$$

Yesterday we found:

$$\underbrace{a_0 + a_1 + 2a_2}_{} + \sum_{n=1}^{\infty} \underbrace{\left(a_{n-1} + a_n + a_{n+1}(n+1)^2 + a_{n+2}(n+1)(n+2) \right)}_{} (x-1)^n = 0$$

for all x

$$\sum b_n (x-a)^n = 0 \text{ for all } x \iff b_n = 0 \text{ for all } n$$

In our case this means that

$$a_0 + a_1 + 2a_2 = 0 \Rightarrow \boxed{a_2 = -\frac{a_0 + a_1}{2}}$$

$$a_{n-1} + a_n + a_{n+1}(n+1)^2 + a_{n+2}(n+1)(n+2) = 0$$

$$\boxed{a_{n+2} = -\frac{a_{n-1} + a_n + a_{n+1}(n+1)^2}{(n+1)(n+2)}}$$

How to use the reccurence relations:

$$a_2 = -\frac{a_0 + a_1}{2}$$

$n=1$

$$a_3 = -\frac{a_0 + a_1 + 4a_2}{2 \cdot 3} = -\frac{a_0 + a_1 - \cancel{4}^{2}\,\frac{a_0 + a_1}{\cancel{2}_{1}}}{6} = \frac{a_0 + a_1}{6}$$

$n=2$

$$a_4 = -\frac{a_1 + a_2 + 9a_3}{3 \cdot 4} = -\frac{a_1 - \frac{a_0 + a_1}{2} + \frac{9}{6}(a_0 + a_1)}{12} = -\frac{1}{12}(a_0 + 2a_1)$$

b) Find the first five terms in the power expansion about $x_0 = 1$ of the solution of the equation (1) satisfying initial conditions $y(1) = 3$, $y'(1) = 1$.

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + a_4(x-1)^4 + \ldots$$

$$y'(x) = a_1 + 2a_2(x-1) + 3a_3(x-1)^2 + \ldots$$

Solve IVP $y(1) = 3$ $y'(1) = 1$

$$y(1) = \boxed{a_0 = 3} \qquad y'(1) = \boxed{a_1 = 1}$$

$$a_2 = -\frac{3+1}{2} = \boxed{-2}$$ (see part (a))

$$a_3 = \frac{3+1}{6} = \boxed{\frac{2}{3}}$$

$$a_4 = -\frac{1}{12}(3+2) = \boxed{-\frac{5}{12}}$$

REMARK 7. *The coefficients a_n of the series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ with $n \geq 2$ are uniquely determined by the first two coefficients a_0 and a_1. Moreover, a_n are expressed linearly in terms of a_0 and a_1 so that*

$$y(x) = a_0 y_1(x) + a_1 y_2(x), \qquad (*$$

where $y_1(x)$ is the solution satisfying the initial conditions $y_1(x_0) = 1$, $y_1'(x_0) = 0$ and $y_2(x)$ is the solution satisfying the initial conditions $y_2(x_0) = 0$, $y_2'(x_0) = 1$.

Let us show the formula (*)

$$y(x) = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + a_4(x-1)^4 + \ldots$$

$$y(x) = a_0 + a_1(x-1) - \frac{a_0 + a_1}{2}(x-1)^2 + \frac{a_0 + a_1}{6}(x-1)^3 - \frac{1}{12}(a_0 + 2a_1)(x-1)^4 + \ldots$$

$$y(x) = a_0 \underbrace{\left(1 - \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 - \frac{1}{12}(x-1)^4 + \ldots\right)}_{y_1(x)} + a_1 \underbrace{\left(x-1 - \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 - \frac{2}{12}(x-1)^4 + \ldots\right)}_{y_2(x)}$$

$$= a_0 y_1(x) + a_1 y_2(x)$$

Initial conditions that y_1 and y_2 satisfy:

$$y_1(1) = 1 \qquad y_2(1) = 0$$

$$y_1'(1) = 0 \qquad y_2'(1) = 1$$

$P(x)y'' + Q(x)y' + R(x)y = 0$

THEOREM 6. *1. If x_0 is an ordinary point of differential equation* (1), *then any solution* $y(x)$ *of* (1) *is analytic at* $x = x_0$, *i.e can be found as a power series* $\sum_{n=0}^{\infty} a_n(x - x_0)^n$.

2. The radius of convergence of this series is at least as large as the minimum of the radii of convergence of the Taylor series at x_0 *of functions* $p(x) := \dfrac{Q(x)}{P(x)}$ *and* $q(x) := \dfrac{R(x)}{P(x)}$.

$R(y(x)) \geqslant \min\{R(p(x)), R(q(x))\}$

EXAMPLE 11. *Determine a lower bound for the radius of convergence of series solutions about each given point of the following equation:*

a) $xy'' + y' + xy = 0$ about $x_0 = 1$ (as in Example 8)

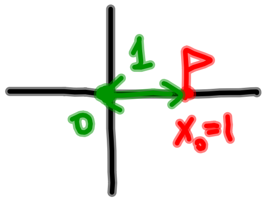
Note that $x_0 = 1$ is ordinary point

$$y'' + \frac{1}{x} y' + y = 0$$

$\Rightarrow$ $p(x) = \frac{1}{x}$

$x=0$ is singular point

$q(x) = 1$ (no singular points $R(q) = \infty$)



$R(p) = 1$

$$R(y) \geq \min\{1, \infty\} \Rightarrow \boxed{R(y) \geq 1}$$

b) $(x^2+2x+2)y''+xy'+4y=0$ about

$$p(x)=\frac{x}{x^2+2x+2}$$

$$q(x)=\frac{4}{x^2+2x+2}$$

Singular points of p and q are zeroes of denominator in this case

$$x^2+2x+2=(x+1)^2+1=0$$

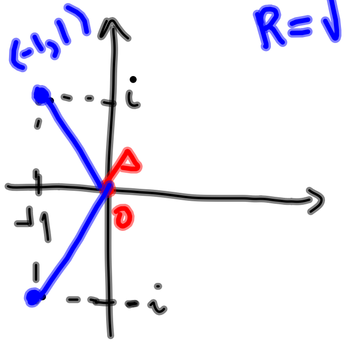
$$x=-1\pm i$$

i) $x_0=0$

$\min\{R(p),R(q)\}=\sqrt{2}$

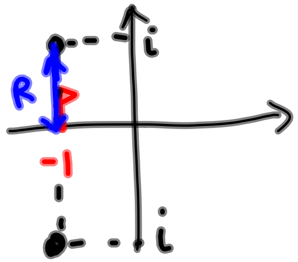
$R=\sqrt{2}=R(p)=R(q)$

$R(y)\geq\sqrt{2}$



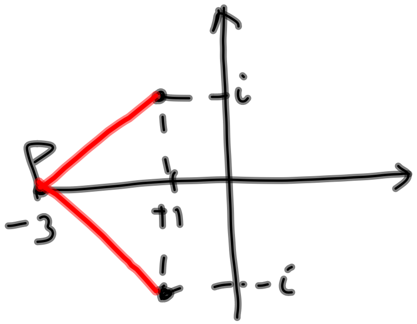
ii) $x_0=-1$

$R(p)=R(q)=1$



$\boxed{R\geq 1}$

iii) $x_0=-3$



$R(p)=R(q)=\sqrt{5}$

$\boxed{R(y)\geq\sqrt{5}}$

EXAMPLE 5. *Given*

$$\sin^2 x\, y'' + x^2 y' + (1-\cos x)y = 0$$

$$p(x) = \frac{x^2}{\sin^2 x} = \left(\frac{x}{\sin x}\right)^2 = \left(\frac{x}{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}\right)^2 = \frac{\cancel{x^2}}{\cancel{x^2}\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right)^2}$$

$$q(x) = \frac{1-\cos x}{\sin^2 x} = \frac{1-\cos x}{1-\cos^2 x} = \frac{\cancel{1-\cos x}}{\cancel{(1-\cos x)}(1+\cos x)} = \frac{1}{1+\cos x}$$

a) is $x_0 = 0$ ordinary or singular?

b) is $x_0 = 2\pi$ ordinary or singular?