Linear HOMOGENEOUS ODE of second order

8. Question: Can the function $y = \sin(t^2)$ be a solution on the interval $(-1, 1)$ of a second order linear homogeneous equation with continuous coefficients?

9. Consider a linear homogeneous ODE

$$y'' + p(t)y' + q(t)y = 0 \tag{2}$$

with coefficients p and q continuous in an interval I.

10. **Superposition Principle**

- Sum $y_1(t) + y_2(t)$ of any two solutions $y_1(t)$ and $y_2(t)$ of (2) is itself a solution.
- A scalar multiple $Cy(t)$ of any solution $y(t)$ of (2) is itself a solution.

COROLLARY 3. *Any* `linear combination` $C_1y_1(t) + C_2y_2(t)$ *of any two solutions* $y_1(t)$ *and* $y_2(t)$ *of (2) is itself a solution.*

$y_1 = \cos t$ $\qquad$ $y_2 = 5\cos t$

$y = C_1 y_1 + C_2 y_2 = C_1 \cos t + 5 C_2 \cos t$

$= C \cos t$

11. `Why Superposition Principle is important?` Once two solutions of a linear homogeneous equation are known, a whole class of solutions is generated by linear combinations of these two.

IVP: $y'' + p(t)\,y' + q(t)\,y = 0 \quad (1)$

$y(t_0) = y_0 \quad , \quad y'(t_0) = v_0 \quad (1^*)$

Assume that $y_1(t)$ and $y_2(t)$ are particular solutions of (1). By Superposition Principle

$$\boxed{y(t) = c_1 y_1(t) + c_2 y_2(t) \quad (2)}$$

is also solution of (1).

(2) is solution of IVP if and only if there exist c_1 and c_2 such that solution (2) satisfies the initial conditions (1^*).

$$\begin{array}{c|l} y(t_0) & c_1 y_1(t_0) + c_2 y_2(t_0) = y_0 \\ \hline y'(t_0) & c_1 y_1'(t_0) + c_2 y_2'(t_0) = v_0 \end{array}$$

By Cramer's Rule

$$\underset{\text{WRONSKIAN}}{W(y_1, y_2)(t_0)} = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} \neq 0$$

then the above system has unique solution (c_1, c_2)

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t) \\ v_0 & y_2'(t) \end{vmatrix}}{W(y_1, y_2)(t_0)} \quad \text{and} \quad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & v_0 \end{vmatrix}}{W(y_1, y_2)(t_0)}$$

12. **WRONSKIAN** of the functions $y_1(t)$ and $y_2(t)$:

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

13. Suppose that $y_1(t)$ and $y_2(t)$ are two differentiable solutions of (2) in the interval I such that $W(y_1, y_2)(t) \neq 0$ somewhere in I, then every solution is a linear combination of $y_1(t)$ and $y_2(t)$.

In other words, the family of solutions $y(t) = C_1 y_1(t) + C_2 y_2(t)$ with arbitrary coefficients C_1 and C_2 includes every solution of (2) if and only if there is a points t_0 where $W(y_1, y_2)$ is not zero. In this case the pair $(y_1(t), y_2(t))$ is called the fundamental set of solutions of (2).

! REMARK 4. Wronskian $W(y_1, y_2)(t)$ (of any two solutions $y_1(t)$ and $y_2(t)$ of (2))either is zero for all t or else is never zero.

For example if $Y_1 = \cos t$, $Y_2 = 5\cos t$ then

$$W(Y_1, Y_2) = \begin{vmatrix} \cos t & 5\cos t \\ -\sin t & -5\sin t \end{vmatrix} = -5\cos t \sin t - (-5\cos t \sin t) = 0$$

$\Downarrow$

$\{y_1, y_2\} = \{\cos t, 5\cos t\}$ is NOT fundamental set

And thus $c_1 \cos t + C_2 \cdot 5\cos t$ is not general solution of ODE of 2nd order.

14. $\underbrace{\text{Confirm that } \sin x \text{ and } \cos x \text{ are solutions of } y'' + y = 0.}$ Then solve the IVP

$$y'' + y = 0, \quad y(\pi) = 0, \quad y'(\pi) = -5$$

$y_1(x) = \sin x \Rightarrow y_1'' + y_1 = (\sin x)'' + \sin x = -\sin x + \sin x = 0$

$y_2(x) = \cos x \Rightarrow y_2'' + y_2 = (\cos x)'' + \cos x = -\cos x + \cos x = 0$

Solve IVP. Determine whether $\{y_1, y_2\}$ is fundamental set ($\Leftrightarrow$ WRONSKIAN $\neq 0$).

$$W(y_1, y_2) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x = -(\sin^2 x + \cos^2 x) = -1 \neq 0$$

$\{y_1, y_2\} = \{\sin x, \cos x\}$ is fundam. set

$\Downarrow$

$y(t) = c_1 \sin t + c_2 \cos t$ is general solution

Solve IVP! Find c_1, c_2 such that

$$\boxed{y(t) = c_1 \sin t + c_2 \cos t}$$

satisfies initial conditions

$$y(\pi) = 0, \quad y'(\pi) = -5$$

$y' = c_1 \cos t - c_2 \sin t$

$y(\pi) = c_1 \cdot 0 + c_2(-1) = 0 \Rightarrow c_2 = 0$

$y'(\pi) = -c_1 + c_2 \cdot 0 = -5 \Rightarrow c_1 = 5$

Solution of IVP! $\boxed{y(t) = 5 \sin t}$

Appendix: Facts from Algebra

1. • FACT 1: **Cramer's Rule** for solving the system of equations

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

The rule says is that if the **`determinant of the coefficient matrix is not zero`**, i.e.

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0,$$

then the system has a unique solution (x, y) given by

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

- FACT 2: If determinant of the coefficient matrix is zero then either there is no solution, or there are infinitely many solutions.
- FACT 3. The homogeneous system of linear equations

$$a_1x + b_1y = 0$$
$$a_1x + b_2y = 0$$

always has the "trivial" solution $(x, y) = (0, 0)$. By Cramer's rule this is the only solution if the determinant of the coefficient matrix is not zero.

- FACT 4: If determinant of the coefficient matrix of homogeneous system of linear equations is zero then there are infinitely many nontrivial solutions $(x, y) \neq (0, 0)$.

2. Use Facts 1-4 to determine if each the following systems of linear equations has one solution, no solutio infinitely many solutions. Then find the solution/s (if any).

(a) $\begin{aligned} 2x + 3y &= 5 \\ x - y &= 4 \end{aligned}$

$$\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} = -2 - 3 = -5 \neq 0$$

$\Downarrow$

the system has a unique solution

$$x = \frac{\begin{vmatrix} 5 & 3 \\ 4 & -1 \end{vmatrix}}{-5} = \frac{-5-12}{-5} = \boxed{\frac{17}{5}} \qquad y = \frac{\begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix}}{-5} = \frac{8-5}{-5} = \boxed{-\frac{3}{5}}$$

$$(x, y) = \left(\frac{17}{5}, -\frac{3}{5}\right)$$

(b) $$\begin{aligned} 2x - 2y &= 4 \\ x - y &= 7 \end{aligned}$$

(c) $$\begin{aligned} 2x - 2y &= 0 \\ 3x + 3y &= 0 \end{aligned}$$

(d) $$\begin{aligned} 2x - 2y &= 0 \\ 3x - 3y &= 0 \end{aligned}$$