Linear HOMOGENEOUS ODE of second order

- 8. Question: Can the function $y = \sin(t^2)$ be a solution on the interval (-1,1) of a second order linear homogeneous equation with continuous coefficients?
- 9. Consider a linear homogeneous ODE

$$y'' + p(t)y' + q(t)y = 0 (2)$$

with coefficients p and q continuous in an interval I.

- 10. Superposition Principle
 - Sum $y_1(t) + y_2(t)$ of any two solutions $y_1(t)$ and $y_2(t)$ of (2) is itself a solution.
 - A scalar multiple Cy(t) of any solution y(t) of (2) is itself a solution.

COROLLARY 3. Any linear combination $C_1y_1(t) + C_2y_2(t)$ of any two solutions $y_1(t)$ and $y_2(t)$ of (2) is itself a solution.

$$y_1 = cost$$

$$y_2 = 5cost$$

$$y = c_1 y_1 + c_2 y_2 = c_1 cost + 5 c_2 cost$$

$$= C cost$$

Title: Feb 3-11:29 PM (Page 1 of 9)

11. Why Superposition Principle is important? Once two solutions of a linear homogeneous equation are known, a whole class of solutions is generated by linear combinations of these two.

IVP:
$$y'' + p(t) y' + q(t) y = 0$$
 (1)

 $y(t_0) = y_0$ (1)

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Assume that $y_1(t)$ end $y_2(t)$ are particular solutions of (1). By Superposition Principle solutions of (1). By Superposition Principle is also solution of (1).

(2) is solution of IVP if and only if there exist C_1 and C_2 such that solution there exist C_1 and C_2 such that solution (t_0) satisfies the initial conditions (t_0) .

 $y(t_0) = C_1 y_1(t_0) + C_2 y_2(t_0) = y_0$
 $y''(t_0) = C_1 y_1(t_0) + C_2 y_2(t_0) = y_0$

By Cramer's Rule

$$W(y_{11}, y_2)(t_0) = \begin{cases} y_1(t_0) & y_2(t_0) \\ y_1(t_0) & y_2(t_0) \end{cases}$$
 $y''(t_0) = \begin{cases} y_1(t_0) & y_2(t_0) \\ y_1(t_0) & y_2(t_0) \end{cases}$
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 $y'''''(t_0) = \begin{cases} y_1(t_0) & y_0 \\ y_1(t_0) & y_0 \end{cases}$

12. WRONSKIAN of the functions $y_1(t)$ and $y_2(t)$:

$$W(y_1, y_2)(t) = \begin{cases} y_1(t) & y_2(t) \\ y_1(t) & y_2(t) \end{cases}$$

Title: Feb 3-11:31 PM (Page 2 of 9)

13. Suppose that $y_1(t)$ and $y_2(t)$ are two differentiable solutions of (2) in the interval I such that $W(y_1, y_2)(t) \neq 0$ somewhere in I, then every solution is a linear combination of $y_1(t)$ and $y_2(t)$.

In other words, the family of solutions $y(t) = C_1y_1(t) + C_2y_2(t)$ with arbitrary coefficients C_1 and C_2 includes every solution of (2) if and only if there is a points t_0 where $W(y_1, y_2)$ is not zero. In this case the pair $(y_1(t), y_2(t))$ is called the **fundamental set** of solutions of (2).

REMARK 4. Wronskian $W(y_1, y_2)(t)$ (of any two solutions $y_1(t)$ and $y_2(t)$ of (2))either is zero for all t or else is never zero.

For example if
$$Y_1 = cost$$
, $Y_2 = 5 cost$ then

For example if $Y_1 = cost$, $Y_2 = 5 cost$ then

 $W(Y_1, Y_2) = \begin{vmatrix} cost & 5 cost \\ -5 cost \end{vmatrix} = -5 cost sint - (-5 cost sint)$
 $W(Y_1, Y_2) = \begin{vmatrix} cost & -5 sint \\ -5 sint \end{vmatrix} = 0$
 $Y_{11}Y_{2} = \{ cost + 5 cost \}$ is NOT fundamental set

And thus $c_1 cost + (2.5 cost)$ is not general solution

of ODE of 2nd order.

Title: Feb 3-11:34 PM (Page 3 of 9)

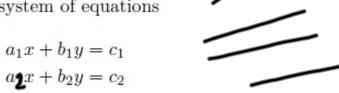
14. Confirm that $\sin x$ and $\cos x$ are solutions of y'' + y = 0. Then solve the IVP y'' + y = 0, $y(\Pi) = 0$, $y'(\Pi) = -5$ $y_1(x) = \sin x$ $= \sum_{i=1}^{y_i+y_i} (\sin x)^{i} + \sin x = -\sin x + \sin x = 0$ $y_2(x) = \cos x = y_2'' + y_2 = (\cos x)'' + \cos x = -\cos x + \cos x = 0$ Solve IVP. Determine whether { 4, 423 is fundamental set ((=> WRONG KIAN =0). $W(y_{11}y_{2}) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = - \sin^{2}x - \cos^{2}x \\ = - (\sin^{2}x + \cos^{2}x) = -1 \neq 0$ $\{y_{11}y_{2}\} = \{\sin x, \cos x\} \text{ is fundem, set}$ y(t) = c, sint + cz est is general solution

Title: Feb 3-11:34 PM (Page 4 of 9)

Solve IVP! Find C1, Cz such that (t) = CISINT + C2 Cost satisfies initial conditions $y'(\pi) = -5$ $y(\pi) = c_1 + c_2(-1) = 0 =)(z=0)$ $y'(\pi) = -c_1 + c_2(-1) = 0 =)(z=0)$ Solution of IVP! | Y(t) = 5 Sint

Appendix: Facts from Algebra





The rule says is that if the determinant of the coefficient matrix is not zero, i.e.

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0,$$

then the system has a unique solution (x, y) given by

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Title: Feb 3-11:35 PM (Page 6 of 9)

- FACT 2: If determinant of the coefficient matrix is zero then either there is no solution, or there are infinitely many solutions.
- FACT 3. The homogeneous system of linear equations

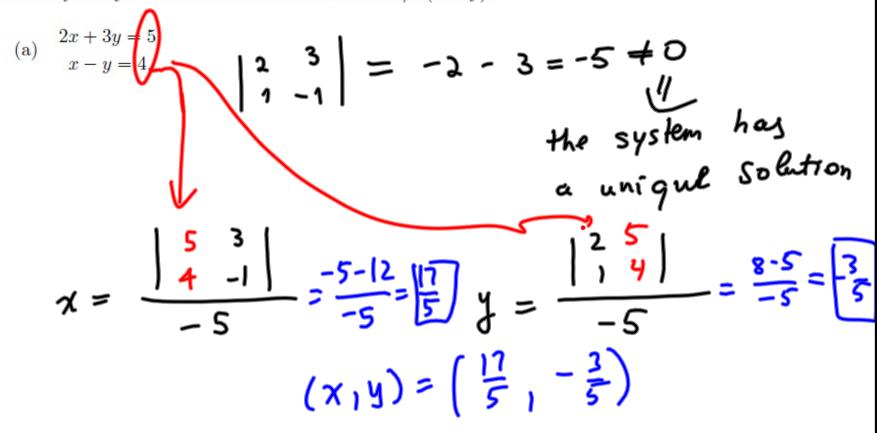
$$a_1x + b_1y = 0$$
$$a_1x + b_2y = 0$$

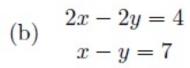
always has the "trivial" solution (x,y) = (0,0). By Cramer's rule this is the only solution if the determinant of the coefficient matrix is not zero.

 FACT 4: If determinant of the coefficient matrix of homogeneous system of linear equations is zero then there are infinitely many nontrivial solutions (x, y) ≠ (0, 0).

Title: Feb 3-11:36 PM (Page 7 of 9)

2. Use Facts 1-4 to determine if each the following systems of linear equations has one solution, no solution infinitely many solutions. Then find the solution/s (if any).





(c)
$$2x - 2y = 0 \\
3x + 3y = 0$$

(d)
$$2x - 2y = 0$$
$$3x - 3y = 0$$