10. Linear homogeneous equations of second order with constant coefficients the case of distinct roots of characteristic equation (sec. 3.1)

1. PROBLEM: Find general solutions of linear homogeneous equation

$$ay'' + by' + cy = 0 \tag{1}$$

   with constant real coefficients a, b, and c.

2. Recall that

   - By Superposition Principle: *Any* `linear combination` $C_1 y_1(t) + C_2 y_2(t)$ *of any two solutions* $y_1(t)$ *and* $y_2(t)$ *of (1) is itself a solution.*
   - The family of solutions $y(t) = C_1 y_1(t) + C_2 y_2(t)$ with arbitrary coefficients C_1 and C_2 includes every solution of (1) if and only if there is a points t_0 where $W(y_1, y_2)$ is not zero. In this case the pair $(y_1(t), y_2(t))$ is called the **fundamental set** of solutions of (1).

3. To find a fundamental set for the equation (1) note that the nature of the equation suggests that it may have solutions of the form

$$y = e^{rt}.$$

Plug in and get so called **characteristic equation** of (1):

$$ar^2 + br + c = 0. \tag{2}$$

$$ay'' + by' + cy = 0$$

$$y = e^{rt}$$

r is a parameter

$$y' = re^{rt}$$

$$y'' = (re^{rt})' = r^2e^{rt}$$

$$cy = ce^{rt}$$

$$+\ by' = bre^{rt}$$

$$ay'' = ar^2e^{rt}$$

$$0 = \underbrace{e^{rt}}_{\neq 0}(ar^2+br+c)$$

$$\Downarrow$$

$$ar^2+br+c=0$$

quadratic equation

Note that the characteristic equation can be determined from its differential equation simply by replacing $y^{(k)}$ with with r^k.

4. *Solve* $y'' - 16y' = 0$.

$$y'' - 16y' = 0$$

1 Char. equation $r^2 - 16r = 0$

$$r(r-16) = 0 \implies r_1 = 0 \implies y_1 = e^{r_1 t} = e^0 = 1$$

$$r_2 = 16 \implies y_2 = e^{r_2 t} = e^{16t}$$

2 $W(y_1, y_2) = \begin{vmatrix} 1 & e^{16t} \\ 0 & 16e^{16t} \end{vmatrix} = 16e^{16t} \neq 0$

3 $\{y_1, y_2\} = \{1, e^{16t}\}$ is fundamental set

$$y(t) = C_1 + C_2 e^{16t}$$

5. `Fact from Algebra:` The quadratic equation $ar^2 + br + c = 0$ has roots

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

discriminant

$D = b^2 - 4ac$

which fall into one of 3 cases:

- two distinct real roots $r_1 \neq r_2$ (in this case $D = b^2 - 4ac > 0$) [section 3.1]
- two complex conjugate roots $r_1 = \overline{r_2}$ (in this case $D = b^2 - 4ac < 0$) [section 3.3]
- two equal real roots $r_1 = r_2$ (in this case $D = b^2 - 4ac = 0$) [section 3.4]

Case 1: Two distinct real roots ($D = b^2 - 4ac > 0$)

6. Two distinct real roots r_1 and r_2 of the characteristic equation give us two solutions

$$\boxed{y_1(t) = e^{r_1 t}, \quad y_2(t) = e^{r_2 t}} \quad (r_1 \neq r_2).$$

$D > 0$

Is this a fundamental set of solutions?

$$W(y_1, y_2) = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = r_2 e^{r_1 t} e^{r_2 t} - r_1 e^{r_1 t} e^{r_2 t}$$

$e^a \cdot e^b = e^{a+b}$

$$= r_2 e^{r_1 t + r_2 t} - r_1 e^{r_1 t + r_2 t} = \underbrace{e^{(r_1 + r_2)t}}_{\neq 0} \underbrace{(r_2 - r_1)}_{\neq 0}$$

$r_1 \neq r_2$