11: Complex roots of the characteristic equation (section 3.3.)

1. **Case 2:** two complex conjugate roots $r_1 = \overline{r_2}$ (in this case $D = b^2 - 4ac < 0$)

2. Recall that the **characteristic equation** of a linear homogeneous equation with constant real coefficients

$$ay'' + by' + cy = 0 \tag{1}$$

$D < 0 \Rightarrow \sqrt{D} = i\sqrt{|D|}$

is

$$ar^2 + br + c = 0. \tag{2}$$

$$D = b^2 - 4ac < 0 \Rightarrow r_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \underbrace{-\frac{b}{2a}}_{=\lambda} \pm i \underbrace{\frac{\sqrt{|D|}}{2a}}_{\mu} =: \lambda + i\mu$$

Then two particular solutions which form a fundamental set are

$\{y_1, y_2\} = \{e^{r_1 t}, e^{r_2 t}\}$

$$y_1 = e^{(\lambda + i\mu)t}, \qquad y_2 = e^{(\lambda - i\mu)t}.$$

$\Downarrow \quad \Downarrow$

$y_1 = \overline{y_2}$

$y_2 = \overline{y_1}$

$$y_1 = e^{\lambda t}\left(\cos(\mu t) + i \sin(\mu t)\right)$$

3. Note that formally there is no difference between this case and Case 1 (two distinct real roots). However in practice we prefer to work with real functions instaed of complex exponentials. By Superposition Principle the following linear combinations are solutions as well

$= \mathrm{Im}(y_1)$

$$\frac{1}{2}(y_1 + y_2) = \underbrace{e^{\lambda t}\cos(\mu t)}, \quad \frac{1}{2i}(y_1 - y_2) = \underbrace{e^{\lambda t}\sin(\mu t)}$$

Note that these solutions are real functions.

$$\frac{1}{2}(y_1 + y_2) = \frac{1}{2}(y_1 + \bar{y}_1) = \mathrm{Re}\, y_1 = e^{\lambda t}\cos(\mu t)$$

4. FACT: $\frac{\mathrm{d}}{\mathrm{d}t}(e^{rt}) = re^{rt}$ for any complex r.

5. Show that $\{e^{\lambda t}\cos(\mu t), e^{\lambda t}\sin(\mu t)\}$ is a fundamental set of solutions, i.e. that general solution of (1) has a form

$$\boxed{y(t) = C_1 e^{\lambda t}\cos(\mu t) + C_2 e^{\lambda t}\sin(\mu t)} \tag{3}$$

Two particular solutions

$$y_1(t) = e^{\lambda t}\cos(\mu t)$$
$$y_2(t) = e^{\lambda t}\sin(\mu t)$$

$$\boxed{e^{\lambda t}\cdot e^{\lambda t} = e^{2\lambda t}}$$

$$W(y_1, y_2) = \begin{vmatrix} e^{\lambda t}\cos(\mu t) & e^{\lambda t}\sin(\mu t) \\ \lambda e^{\lambda t}\cos(\mu t) - \mu e^{\lambda t}\sin \mu t & \lambda e^{\lambda t}\sin(\mu t) + \mu e^{\lambda t}\cos(\mu t) \end{vmatrix}$$

$$= e^{2\lambda t}\left[\cancel{\lambda\cos(\mu t)\sin(\mu t)} + \mu\cos^2(\mu t) - \cancel{\lambda\sin(\mu t)\cos\mu t} + \mu\sin^2(\mu t)\right]$$

$$= e^{2\lambda t}\mu\,\underbrace{(\cos^2(\mu t) + \sin^2(\mu t))}_{=1} = \underbrace{e^{2\lambda t}}_{\neq 0}\mu \neq 0$$

because $r_{1,2} = \lambda \pm i\mu$ $(D < 0)$

6. Solve the following two differential equations which are important in applied mathematics:

$$y'' + \omega^2 y = 0 \quad \text{and} \quad y'' - \omega^2 y = 0,$$

where ω is a real positive constant.

$y'' + \omega^2 y = 0$

$r^2 + \omega^2 = 0$

$r^2 = -\omega^2 \Rightarrow r_{1,2} = \sqrt{-\omega^2} = \boxed{\pm i\omega}$

$r = \lambda \pm i\mu$

$\lambda = 0, \quad \mu = \omega$

Fundam. set

$\{e^{\lambda t}\cos\mu t, e^{\lambda t}\sin\mu t\} = \{e^{0t}\cos(\omega t), e^{0t}\sin(\omega t)\}$

$= \{\cos(\omega t), \sin(\omega t)\}$

$$\boxed{y(t) = C_1\cos(\omega t) + C_2\sin(\omega t)}$$

$y'' - \omega^2 y = 0$

$r^2 - \omega^2 = 0 \Rightarrow r_{1,2} = \pm\omega$

2 real distinct roots

Fundam. set

$\{e^{r_1 t}, e^{r_2 t}\} = \{e^{\omega t}, e^{-\omega t}\}$

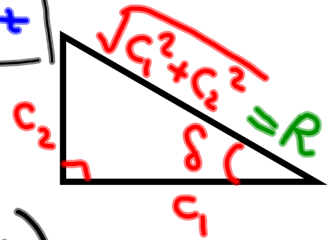
$$\boxed{y(t) = c_1 e^{\omega t} + c_2 e^{-\omega t}}$$

7. Alternative form of solution (3):

δ delta

$ay'' + by' + cy = 0$ where $D = b^2 - 4ac < 0 \Rightarrow r_{1,2} = \lambda \pm i\mu$

General solution $\boxed{y(t) = c_1 e^{\lambda t}\cos(\mu t) + C_2 e^{\lambda t}\sin\mu t}$



$$y(t) = e^{\lambda t}(c_1\cos(\mu t) + c_2\sin(\mu t))\frac{\sqrt{c_1^2+c_2^2}}{\sqrt{c_1^2+c_2^2}}$$

$$y(t) = e^{\lambda t}\sqrt{c_1^2+c_2^2}\left(\frac{c_1}{\sqrt{c_1^2+c_2^2}}\cos(\mu t) + \frac{c_2}{\sqrt{c_1^2+c_2^2}}\sin(\mu t)\right)$$

$$y(t) = e^{\lambda t} R \left(\cos\delta\cos(\mu t) + \sin\delta\sin(\mu t)\right)$$

Remind $\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha-\beta) = \cos(\beta-\alpha)$

$$\boxed{y(t) = e^{\lambda t} R\cos(\mu t - \delta)}$$

where $R = \sqrt{c_1^2 + c_2^2}$

$\cos\delta = \frac{c_1}{R}$, $\sin\delta = \frac{c_2}{R}$

$\tan\delta = \frac{c_2}{c_1}$ ← what quadrant

$$y(t) = e^{\lambda t} R\cos(\mu t - \delta), \tag{4}$$

where

$$R = \sqrt{C_1^2 + C_2^2}, \quad \cos\delta = \frac{C_1}{\sqrt{C_1^2+C_2^2}} = \frac{C_1}{R}, \quad \sin\delta = \frac{C_2}{\sqrt{C_1^2+C_2^2}} = \frac{C_2}{R}.$$

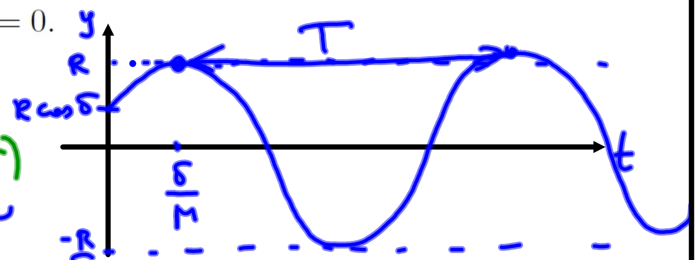
Note that $\tan\delta = C_2/C_1$.

8. Application: Mechanical unforced vibration: a mass hanging from a spring (more details in Section 3.7).

$$y(t) = e^{\lambda t} R \cos(\mu t - \delta)$$

- $\lambda = 0$ corresponds to **undamped** free vibration (simple harmonic motion)
- $\lambda < 0$ corresponds to **damped** free vibration
- R is called the **amplitude** of the motion

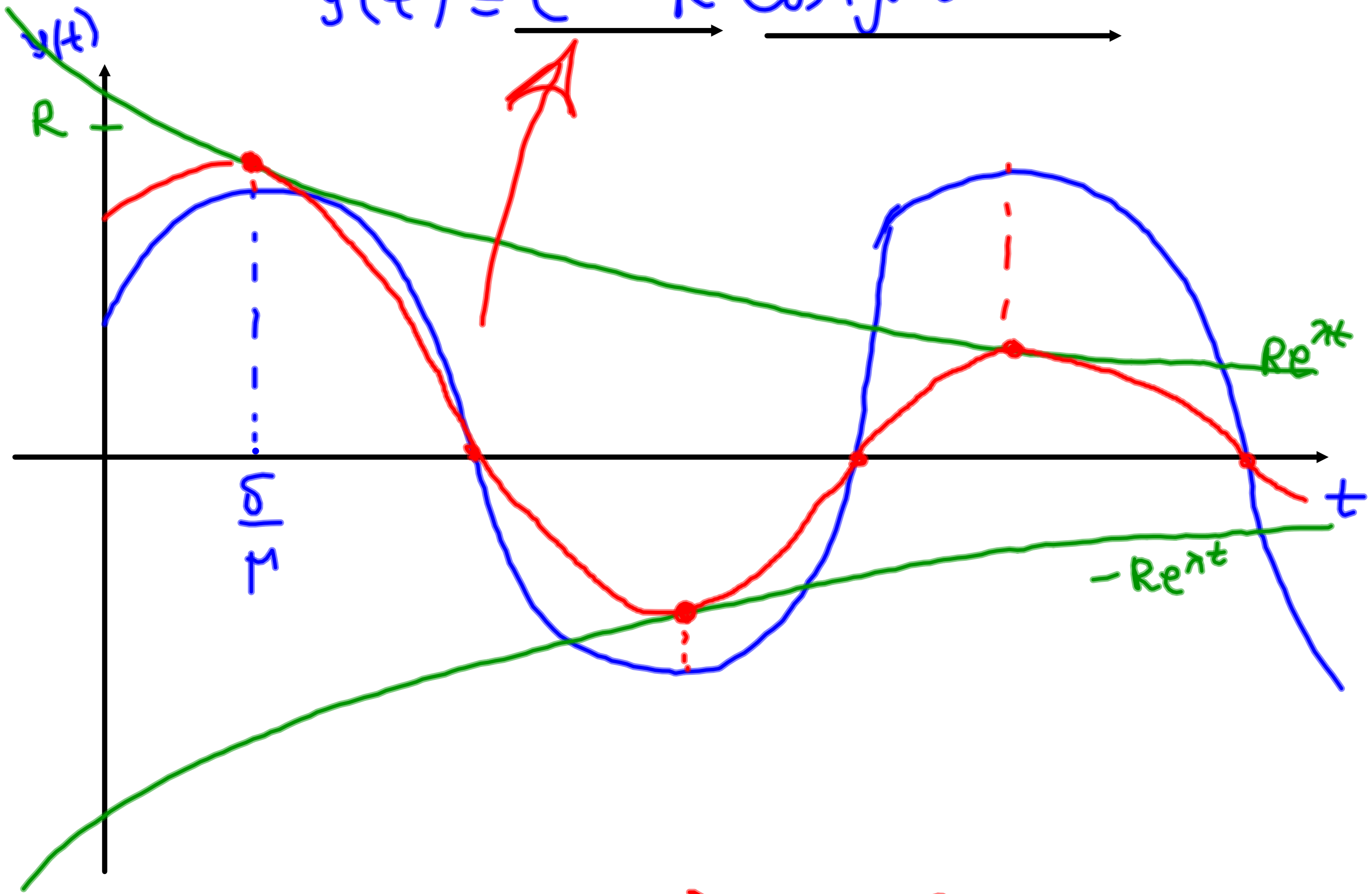
- δ is called the **phase**, or phase angle, and measures the displacement of the wave from its normal position corresponding to $\delta = 0$.
- $T = \dfrac{2\pi}{\mu}$ is the **period** of the motion.



$$\lambda = 0 \Rightarrow y(t) = R\cos(\underbrace{\mu t - \delta}_{=0}) \Rightarrow t = \frac{\delta}{\mu}$$

$\lambda < 0$

$$y(t) = e^{\lambda t} R \cos(\mu t - \delta)$$



$\lambda < 0 \Rightarrow y(t) \to 0$ as t increases

9. Consider

$$y'' + 2y' + 3y = 0. \tag{5}$$

(a) Find general solution.

$$r^2 + 2r + 3 = 0$$

$$D = 2^2 - 4\cdot 3 = -8 < 0$$

2 complex conjugate roots

$$\sqrt{D} = \sqrt{-8} = i\sqrt{8} = i2\sqrt{2}$$

$$r_{1,2} = \frac{-2 \pm i2\sqrt{2}}{2} = -1 \pm i\sqrt{2} =: \lambda + i\mu$$

$$\lambda = -1, \qquad \mu = \sqrt{2}$$

Fundamental set

$$\{e^{\lambda t}\cos(\mu t), e^{\lambda t}\sin(\mu t)\} = \{e^{-t}\cos(\sqrt{2}t), e^{-t}\sin(\sqrt{2}t)\}$$

$$y(t) = c_1 e^{-t}\cos(\sqrt{2}t) + c_2 e^{-t}\sin(\sqrt{2}t)$$

OR

$$\boxed{y(t) = e^{-t}(c_1\cos(\sqrt{2}t) + c_2\sin(\sqrt{2}t))}$$

(b) Find solution of (5) subject to the initial conditions

$$y(0) = 2, \quad y'(0) = 1.$$

$\boxed{y(t) = e^{-t}\left(c_1 \cos(\sqrt{2}\,t) + c_2 \sin(\sqrt{2}\,t)\right)}$

$2 = y(0) = c_1 \Rightarrow \boxed{c_1 = 2}$

$y'(t) = -e^{-t}\left(c_1 \cos(\sqrt{2}t) + c_2 \sin\sqrt{2}t\right) + e^{-t}\left(-\sqrt{2}\,c_1 \sin(\sqrt{2}t) + \sqrt{2}\,c_2 \cos(\sqrt{2}t)\right)$

$1 = y'(0) = -2 + 0 + 0 + \sqrt{2}\,c_2 \Rightarrow 1 = -2 + \sqrt{2}\,c_2$

$\Rightarrow \boxed{c_2 = \frac{3}{\sqrt{2}}}$

$$\boxed{y(t) = e^{-t}\left(2\cos(\sqrt{2}t) + \frac{3}{\sqrt{2}}\sin(\sqrt{2}t)\right)}$$

(c) Sketch the graph of the solution of IVP from (b) and describe its behavior as t increases.

$$y(t) = e^{-t}\left(2\cos(\sqrt{2}t) + \frac{3}{\sqrt{2}}\sin(\sqrt{2}t)\right)$$

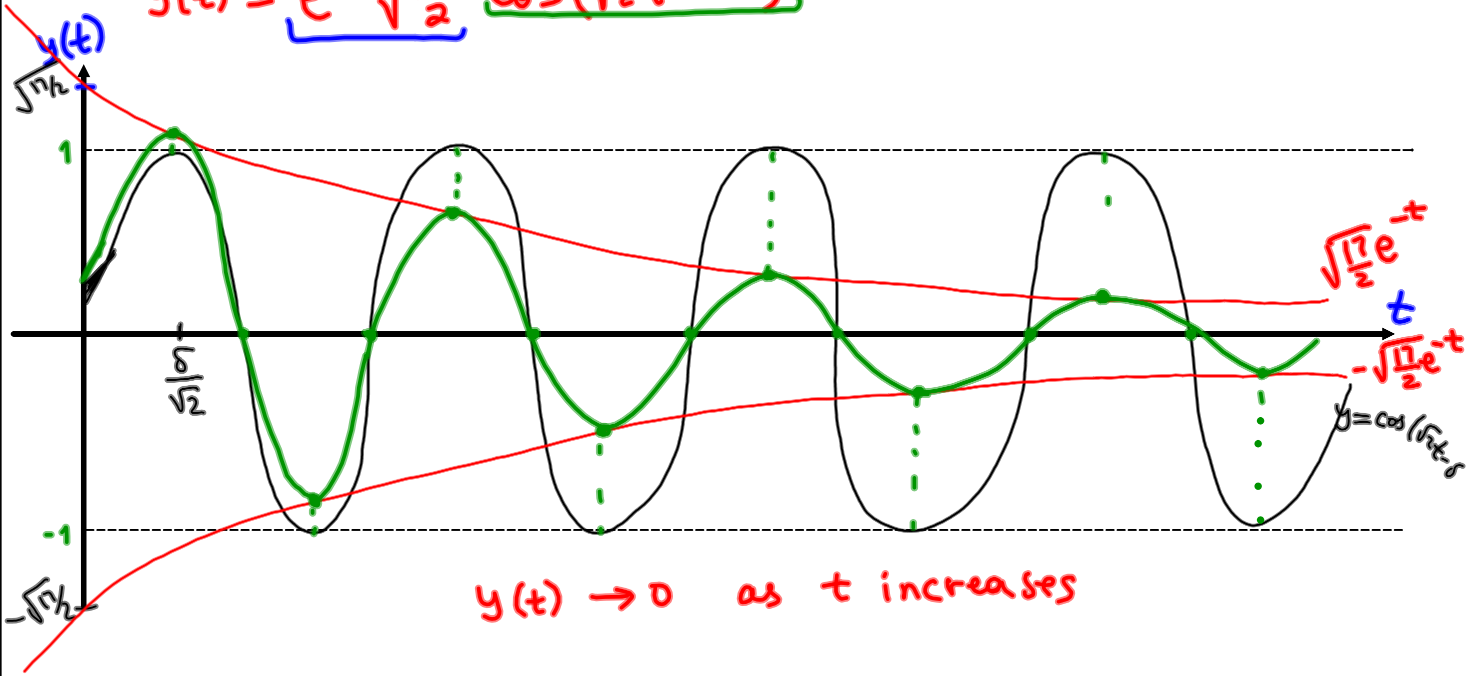
Rewrite $y(t)$ in alternative form $y(t) = e^{\lambda t} R\cos(\mu t - \delta)$

In our case $\lambda = -1$, $\mu = \sqrt{2}$

$c_1 = 2$, $c_2 = \frac{3}{\sqrt{2}}$, $R = \sqrt{c_1^2 + c_2^2} = \sqrt{4 + \frac{9}{2}} = \sqrt{\frac{17}{2}}$

$\delta = \arctan \frac{c_2}{c_1} = \arctan \frac{3}{2\sqrt{2}}$, $0 < \delta < \pi/2$

$$y(t) = \underbrace{e^{-t}\sqrt{\frac{17}{2}}}\ \underline{\cos(\sqrt{2}t - \delta)}$$



$y(t) \to 0$ as t increases