31: Power Series, Taylor Series and Analytic Functions (section 5.1)

DEFINITION 1. A power series about \( x = x_0 \) (or centered at \( x = x_0 \)), or just power series, is any series that can be written in the form

\[
\sum_{n=0}^{\infty} a_n (x - x_0)^n,
\]

where \( x_0 \) and \( a_n \) are numbers.

DEFINITION 2. A power series \( \sum_{n=0}^{\infty} a_n (x - x_0)^n \) is said to converge at a point \( x \) if the limit

\[
\lim_{m \to \infty} \sum_{n=0}^{m} a_n (x - x_0)^n
\]

exists and is finite.

REMARK 3. A power series \( \sum_{n=0}^{\infty} a_n (x - x_0)^n \) always converges at \( x = x_0 \).

EXAMPLE 4. For what \( x \) does the power series \( \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots \) converge?

\[
\sum_{n=0}^{m} x^n
\]

If \( |x| < 1 \)

If \( |x| > 1 \)

Absolute Convergence: The series \( \sum_{n=0}^{\infty} a_n (x - x_0)^n \) is said to converge absolutely at \( x \) if

\[
\sum_{n=0}^{\infty} |a_n| |x - x_0|^n
\]

converges.

If a series converges absolutely then it converges (but in general not vice versa).

EXAMPLE 5. The series \( \sum_{n=1}^{\infty} \frac{x^n}{n} \) converges at \( x = -1 \), but it doesn’t converge absolutely there:

\[
1 - \frac{1}{2} + \frac{1}{3} - \ldots = \ln 2
\]

but the series of absolute values is the so-called harmonic series

\[
1 + \frac{1}{2} + \frac{1}{3} + \ldots
\]

and it is divergent.

Fact: If the series \( \sum_{n=0}^{\infty} a_n (x - x_0)^n \) converges absolutely at \( x = x_1 \) then it converges absolutely for all \( x \) such that \( |x - x_0| < |x_1 - x_0| \).
This immediately implies the following:

**THEOREM 6.** For a given power series \( \sum_{n=0}^{\infty} a_n(x - x_0)^n \) there are only 3 possibilities:

1. The series converges only for \( x = x_0 \).
2. The series converges for all \( x \).
3. There is \( R > 0 \) such that the series converges if \( |x - x_0| < R \) and diverges if \( |x - x_0| > R \). We call such \( R \) the **radius of convergence**.

**REMARK 7.** In case 1 of the theorem we say that \( R = 0 \) and in case 2 we say that \( R = \infty \).

**EXAMPLE 8.** What is the radius of convergence of the geometric power series \( \sum_{n=0}^{\infty} x^n \)?

**How to find Radius of convergence:** If \( a_n \neq 0 \) for any \( n \) and \( \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| \) exists, then

\[
R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|
\]

More generally,

\[
R = \frac{1}{\lim_{n \to \infty} \sqrt[n]{a_n}}
\]

the Cauchy-Hadamard formula.

**EXAMPLE 9.** Find the radius of convergence of the power series \( \sum_{n=0}^{\infty} \frac{n^2}{3^n} (x + 1)^n \).

**The Taylor series for** \( f(x) \) **about** \( x = x_0 \)

Assume that \( f \) has derivatives of any order at \( x = x_0 \). Then for any \( m \)

\[
f(x) = \sum_{n=0}^{m} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n + \frac{f^{(m+1)}(c)}{(m+1)!} (x - x_0)^{m+1}
\]

where \( c \) is a number between \( x \) and \( x_0 \). The remainder converges to zero at least as fast as \( (x - x_0)^{m+1} \) when \( x \to x_0 \).
Formally we can consider the following power series:

\[ \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n, \]

the Taylor series of the function \( f \) about \( x_0 \).

The Taylor series may converge and may not converge in a neighborhood of \( x_0 \) and even if it converges for any \( x \) close to \( x_0 \) it may not converge to \( f(x) \).

**DEFINITION 10.** The function \( f \) is called **analytic** at the point \( x_0 \) if there exist a power series

\[ \sum_{n=0}^{\infty} a_n (x - x_0)^n \]

such that

\[ f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \]

for all \( x \) sufficiently close to \( x_0 \).

In this case, one can show that \( a_n \) must be equal to \( \frac{f^{(n)}(x_0)}{n!} \). This implies:

*The function \( f \) is analytic at the point \( x_0 \) if its Taylor series

\[ \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n \]

converges to \( f(x) \) for all \( x \) sufficiently close to \( x_0 \).*

**EXAMPLE 11.** From Example 3 it follows that \( f(x) = \frac{1}{1-x} \) is analytic at 0. In general, it is analytic at any \( x_0 \neq 1 \)

**EXAMPLE 12.** More generally, any rational function \( f(x) = \frac{Q(x)}{P(x)} \), where \( P(x) \) and \( Q(x) \) are polynomials without common linear factors (the latter can be always assumed, because the common factors can be canceled) is analytic at all points except zeros of the denominator \( P(x) \).

**EXAMPLE 13.** The functions \( e^x \), \( \sin x \) and \( \cos x \) are analytic at any \( x \). Here are their Taylor series at 0:

\[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \]

\[ \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \]

\[ \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \]
REMARK 14. Not any function having derivatives of any order at any point is analytic. For example, take

\[ f(x) = \begin{cases} 
  e^{-\frac{1}{x}}, & x > 0 \\
  0, & x \leq 0 
\end{cases} \]

**Term by term differentiation** If \( f \) is analytic at a point \( x_0 \), i.e. \( f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \) for \( x \) sufficiently close to \( x_0 \), then \( f'(x) \) is also analytic at \( x_0 \) and

\[ f'(x) = \sum_{n=1}^{\infty} na_n (x - x_0)^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}(x-x_0)^n \]

In other words, the derivative of a (convergent) power series is obtained by term by term differentiation of the series.

**EXAMPLE 15.** What is the Taylor expansion of \( f''(x) \)?

\[
 f''(x) = \left( \sum_{n=0}^{\infty} (n+1)a_{n+1}(x-x_0)^n \right)' = \sum_{n=1}^{\infty} (n+1)a_{n+1}(x-x_0)^{n-1} = \\
 \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(x-x_0)^n .
\]