SOLUTIONS - MATH 171H EXAM 1 - FALL 2015

1. (a) The slope is $\frac{3}{-2} = -\frac{3}{2}$, so an equation is $y = -\frac{3}{2}x + 2$.

(b) $x^2 + y^2 = 1$ (c) $\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$, or $\frac{x^2}{16} + \frac{y^2}{9} = 1$

2. (a) for every $\varepsilon > 0$, there is a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$.

(b) Scratch work: $|2x^2 + 1 - 3| = |2x^2 - 2| = 2|x+1|\,|x-1|$

Suppose $\delta \le 1$. Then, whenever $0 < |x-1| < \delta$, we must have

$$|x-1| < 1, \text{ or } -1 < x-1 < 1.$$

Adding 2, we find $1 < x+1 < 3$, so $|x+1| < 3$. In the above expression $2|x+1|\,|x-1|$, the factor $2|x+1| < 2 \cdot 3 = 6$ in this case. So we will want $\delta \le \frac{\varepsilon}{6}$ in the proof.

Proof: For any $\varepsilon > 0$, let $\delta = \min\{1, \frac{\varepsilon}{6}\}$. Then

$$|2x^2 + 1 - 3| = 2|x+1|\,|x-1| < 2 \cdot 3 \cdot \delta = 6\delta \le 6 \cdot \frac{\varepsilon}{6} = \varepsilon$$

whenever $0 < |x-1| < \delta$.

3. (a) $$\lim_{x\to 4} \frac{x-4}{2-\sqrt{x}} \cdot \frac{2+\sqrt{x}}{2+\sqrt{x}} = \lim_{x\to 4} \frac{(x-4)(2+\sqrt{x})}{4-x} = \lim_{x\to 4} \frac{(x-4)(2+\sqrt{x})}{-(x-4)}$$
$$= \lim_{x\to 4}\left(-(2+\sqrt{x})\right) = -4.$$

(b) $$\lim_{x\to\infty} \frac{\sqrt[3]{x^3-1}}{1+3x} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\sqrt[3]{(x^3-1)\cdot\frac{1}{x^3}}}{\frac{1}{x}+3} = \lim_{x\to\infty} \frac{\sqrt[3]{1-\frac{1}{x^3}}}{\frac{1}{x}+3} = \frac{\sqrt[3]{1-0}}{0+3} = \frac{1}{3}$$

(c) $-1 \le \sin\left(\frac{x+1}{x}\right) \le 1$ for all x, so

$-|x|^3 \le x^3 \sin\left(\frac{x+1}{x}\right) \le |x|^3$ for all x. (If $x<0$, multiplying by x^3 changes the order of inequality.)

Note that $\lim_{x\to 0}\left(-|x|^3\right) = 0$ and $\lim_{x\to 0}\left(|x|^3\right) = 0$, so by the Squeeze Theorem, $\lim_{x\to 0} x^3 \sin\left(\frac{x+1}{x}\right) = 0$.

4. (a) $\lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

(b) $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} \cdot \dfrac{x^2(x+h)^2}{x^2(x+h)^2}$

$= \lim_{h\to 0} \dfrac{x^2-(x+h)^2}{hx^2(x+h)^2} = \lim_{h\to 0} \dfrac{x^2-x^2-2xh-h^2}{hx^2(x+h)^2}$

$= \lim_{h\to 0} \dfrac{-2xh-h^2}{hx^2(x+h)^2} = \lim_{h\to 0} \dfrac{-2x-h}{x^2(x+h)^2} = \dfrac{-2x}{x^4} = -\dfrac{2}{x^3}$

5. (a) False. Counterexample: $\vec{a} = (1,0)$, $\vec{b} = (0,1)$. Then $\vec{a}\cdot\vec{b} = 0$, but these vectors are not parallel. (Many other counterexamples are possible. In fact, $\vec{a}\cdot\vec{b} = 0$ implies $\vec{a}$ is orthogonal to $\vec{b}$.)

(b) False. Counterexample: $f(x) = \begin{cases} -1, & \text{if } x \le \frac{1}{2} \\ 1, & \text{if } x > \frac{1}{2} \end{cases}$

Then $f^2(x) = 1$ for all x, so f^2 is continuous on $[0,1]$, but f is not.

(c) True. This is a composition of two continuous functions, which is continuous:
$f^2(x) = (g\circ f)(x)$ where $g(x) = x^2$.

(d) True. Let $f(x) = x^5 + x + 1$, a continuous function. Note that
$f(-1) = -1$ and $f(1) = 3$.
By the Intermediate Value Theorem, there is some c in $(-1,1)$ for which $f(c) = 0$.