Varieties for Representations and Tensor Categories

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$$\text{ALGEBRA} \rightsquigarrow \text{GEOMETRY}$$

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representations of rings/ objects in categories $\rightsquigarrow$ support varieties

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$\searrow$ cohomology

$$\begin{tikzcd}
\text{ALGEBRA} \arrow[rr, squiggly] & & \text{GEOMETRY} \\
\text{representations of rings/ objects in categories} \arrow[rr, squiggly] \arrow[dr, squiggly] & & \text{support varieties} \\
& \text{cohomology} \arrow[ur, squiggly] &
\end{tikzcd}$$

Finite tensor categories

Definition A **finite tensor category** $\mathcal{C}$ is a locally finite k-linear abelian category with finitely many simple objects, enough projectives, and a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ satisfying some conditions. There is a **unit object** 1 that is simple, and every object has both left and right duals.

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Examples

(1) G - finite group, k - field, $\mathcal{C} = kG$-mod, the category of finitely generated kG-modules, $\otimes = \otimes_k$:

For X, Y in $\mathcal{C}$, $X \otimes Y$ is in $\mathcal{C}$ where

$g \cdot (x \otimes y) = (g \cdot x) \otimes (g \cdot y)$ for all $g \in G$, $x \in X$, $y \in Y$

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(2) More generally, $\mathcal{C} = A$-mod for a finite dimensional Hopf algebra A (definition summarized next)

Hopf algebras

A **Hopf algebra** is an algebra A over a field k together with algebra homs. $\Delta : A \to A \otimes A,\ \varepsilon : A \to k$ and an algebra anti-hom. $S : A \to A$ satisfying some conditions.

Hopf algebras include:
- group algebras kG ($\Delta(g) = g \otimes g$ for all $g \in G$)
- universal enveloping algebras of Lie algebras $U(\mathfrak{g})$,
  restricted Lie algebras ($\Delta(x) = x \otimes 1 + 1 \otimes x$ for all $x \in \mathfrak{g}$)
- quantum groups $U_q(\mathfrak{g})$, small quantum groups $u_q(\mathfrak{g})$

Their categories of modules are examples of tensor categories:
If M, N are A-modules, then $M \otimes N$ is an A-module via Δ;
set $\mathbf{1} = k$, an A-module via the augmentation map $\varepsilon : A \to k$

Cohomology

$\mathcal{C}$ - finite tensor category with tensor product $\otimes$ and unit object $\mathbf{1}$

Notation $\mathsf{H}^*(\mathcal{C}) := \mathsf{Ext}^*_{\mathcal{C}}(\mathbf{1},\mathbf{1}) = \bigoplus_{n\geq 0} \mathsf{Ext}^n_{\mathcal{C}}(\mathbf{1},\mathbf{1})$

where $\mathsf{Ext}^n_{\mathcal{C}}(\mathbf{1},\mathbf{1})$ consists of equivalence classes of n**-extensions**

$$\mathbf{1} \to X_{n-1} \to \cdots \to X_1 \to X_0 \to \mathbf{1}$$

of objects in $\mathcal{C}$.

Products on Cohomology

$\mathrm{H}^*(\mathcal{C}) := \mathrm{Ext}^*_A(\mathbf{1}, \mathbf{1})$ is an algebra with binary operations

(1) **Yoneda splice**:

$$(\mathbf{1} \to X_{m-1} \to \cdots \to X_0 \to \mathbf{1}) \times (\mathbf{1} \to Y_{n-1} \to \cdots \to Y_0 \to \mathbf{1})$$

$$= (\mathbf{1} \to X_{m-1} \to \cdots \to X_0 \to Y_{n-1} \to \cdots \to Y_0 \to \mathbf{1})$$

(2) **cup product**, the total complex of the tensor product of complexes:

$$\mathbf{1} \to X_{m-1} \oplus Y_{n-1} \to \cdots \to (X_1 \otimes Y_0) \oplus (X_0 \otimes Y_1) \to X_0 \otimes Y_0 \to \mathbf{1}$$

Properties

• These binary operations on $\mathrm{H}^*(\mathcal{C})$ coincide and are graded commutative by the **Hilton-Eckmann argument** (Suárez-Álvarez): The cup product is an algebra homomorphism with respect to Yoneda splice (and they share the same unit).

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Conjecture (Etingof-Ostrik 2004)
If $\mathcal{C}$ is a finite tensor category, then $\mathrm{H}^*(\mathcal{C})$ is finitely generated, and $\mathrm{H}^*(X)$ is a finitely generated $\mathrm{H}^*(\mathcal{C})$-module.

Status of the conjecture

$\mathrm{H}^*(\mathcal{C})$ is known to be finitely generated etc. in case:

• $\mathcal{C} = A$-mod for a fin. dim. cocommutative Hopf algebra A in positive characteristic (Friedlander-Suslin 1997, generalizing work of Golod 1959, Venkov 1959, Evens 1960, Friedlander-Parshall 1983)

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- $\mathcal{C} = A$-mod for many other classes of Hopf algebras (Gordon 2000, Mastnak-Pevtsova-Schauenburg-W 2010, Vay-Ştefan 2016, Drupieski 2016, Erdmann-Solberg-Wang 2018, Nguyen-Wang-W 2018, Friedlander-Negron 2018, Negron-Plavnik 2018, Angiono-Andruskiewitsch-Pevtsova-W, $\ldots$)

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Object X of $\mathcal{C}$ $\rightsquigarrow$ $\mathsf{H}^*(\mathcal{C})$-module $\mathsf{H}^*(X) := \mathsf{Ext}^*_{\mathcal{C}}(X, X)$
(and the maximal ideal spectrum of the quotient of $\mathsf{H}^*(\mathcal{C})$ by its annihilator)

Varieties for tensor categories: details

From now on let $\mathcal{C}$ be a finite tensor category for which $\mathsf{H}^*(\mathcal{C})$ is a finitely generated graded commutative algebra over the field k, and $\mathsf{H}^*(X) := \mathsf{Ext}^*_{\mathcal{C}}(X, X)$ is a finitely generated $\mathsf{H}^*(\mathcal{C})$-module for each object X of $\mathcal{C}$.

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See also Buan-Krause-Snashall-Solberg, arXiv 2017, for tensor triangulated categories

Varieties for tensor categories: example

$\mathcal{C} = kG$-mod, where G is a cyclic group of prime order p and k has characteristic p:

$\mathsf{H}^*(\mathcal{C})$ is essentially $k[x]$, so $V_{\mathcal{C}}(k)$ is a line.

More generally if X is the indecomposable kG-module with $\dim_k(X) = n$ and $n < p$, then $V_{\mathcal{C}}(X)$ is a line.

Properties of support varieties (Bergh-Plavnik-W, arXiv 2019)

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(2) $V_{\mathcal{C}}(X \oplus Y) = V_{\mathcal{C}}(X) \cup V_{\mathcal{C}}(Y)$

(3) If $0 \to X_1 \to X_2 \to X_3 \to 0$ is a short exact sequence, then $V_{\mathcal{C}}(X_i) \subseteq V_{\mathcal{C}}(X_j) \cup V_{\mathcal{C}}(X_l)$ whenever $\{i, j, l\} = \{1, 2, 3\}$

Complexity

Let $X_1, \dots, X_r$ be the (iso classes) of simple objects of $\mathcal{C}$ and let

$$(N_X)_{ij} = ([X \otimes X_i : X_j])_{ij}$$

for each object X, where $[X \otimes X_i : X_j]$ is the multiplicity of X_j in a Jordan-Hölder series of $X \otimes X_i$. The **Frobenius-Perron dimension** of X, denoted $\mathsf{FPdim}(X)$, is the largest nonnegative real eigenvalue of N_X.

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The **complexity** $\mathrm{cx}_{\mathcal{C}}(X)$ of an object X is the rate of growth of a minimal projective resolution $P_\bullet$ of X, $\cdots \to P_2 \to P_1 \to P_0 \to X \to 0$, as measured by Frobenius-Perron dimensions.

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Theorem (Bergh-Plavnik-W, arXiv 2019) Let X be an object of $\mathcal{C}$. Then $\mathrm{cx}_{\mathcal{C}}(X) = \dim V_{\mathcal{C}}(X)$.

Complexity: proof idea

Theorem (Bergh-Plavnik-W, arXiv 2019) Let X be an object of $\mathcal{C}$. Then $\mathsf{cx}_{\mathcal{C}}(X) = \dim V_{\mathcal{C}}(X)$.

Proof idea

$[P_n : P(X_i)] = \dim_k \mathrm{Hom}_{\mathcal{C}}(P_n, X_i)$ by standard arguments

$= \dim_k \mathrm{Ext}^n_{\mathcal{C}}(X, X_i)$ since $P_\bullet$ is minimal, and so

$\mathrm{FPdim}(P_n) = \sum_i \mathrm{FPdim}(P(X_i)) \cdot \dim_k(\mathrm{Ext}^n_{\mathcal{C}}(X, X_i))$,

which implies $\mathsf{cx}_A(X) \leq \dim V_{\mathcal{C}}(X)$.

Further analysis yields $\mathsf{cx}_A(X) \geq \dim V_{\mathcal{C}}(X)$.

Carlson's L_ζ-objects and realization

Let $\Omega^n(\mathbf{1})$ be the nth syzygy of $\mathbf{1}$, and $\zeta \in \mathsf{H}^n(\mathcal{C}) \cong \mathsf{Hom}_{\mathcal{C}}(\Omega^n(\mathbf{1}), \mathbf{1})$, nonzero. Since $\mathbf{1}$ is simple, there is an object L_ζ and a short exact sequence

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$$V_{\mathcal{C}}(L_\zeta \otimes X) = V_{\mathcal{C}}(L_\zeta) \cap V_{\mathcal{C}}(X),$$

and $V_{\mathcal{C}}(L_\zeta) = Z(\zeta) := \mathrm{Max}(\mathrm{H}^*(\mathcal{C})/(\zeta))$.

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Corollary If V is any conical subvariety of $V_{\mathcal{C}}(\mathbf{1})$, then there is an object X of $\mathcal{C}$ with $V_{\mathcal{C}}(X) = V$.

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Proof of corollary idea
$V = Z(\zeta_1, \dots, \zeta_t)$ for some $\zeta_1, \dots, \zeta_t$, so set $X = L_{\zeta_1} \otimes \cdots \otimes L_{\zeta_t}$.

Connectedness

Theorem (Bergh-Plavnik-W, arXiv 2019) The projective support variety of an indecomposable object is connected.
Precisely: Let X be an object of $\mathcal{C}$ for which $V_{\mathcal{C}}(X) = V_1 \cup V_2$ where V_1, V_2 are conical subvarieties of $V_{\mathcal{C}}(X)$ with $V_1 \cap V_2 = 0$. Then $X \cong X_1 \oplus X_2$ for some objects X_1, X_2 with $V_{\mathcal{C}}(X_i) = V_i$.

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Proof idea Induction on $\dim V_1 + \dim V_2$:
The objects L_ζ and the property $V_{\mathcal{C}}(L_\zeta \otimes Y) = Z(\zeta) \cap V_{\mathcal{C}}(Y)$ are used to reduce this total dimension.

Tensor product property

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Remark This is known to be an equality when $\mathcal{C} = A$-mod for a cocommutative Hopf algebra A (Friedlander-Pevtsova 2005) or when A is a quantum elementary abelian group, i.e. a tensor product of Taft algebras (Pevtsova-W 2015).

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In general, it is only known to be an equality when $X = L_\zeta$ for some ζ. This fact alone is already very useful in proofs, as we have seen.

It is known **not** to be an equality for some modules of some noncocommutative Hopf algebras (Benson-W 2014, Plavnik-W 2018).

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(2) Is $V_{\mathcal{C}}(X \otimes Y) = V_{\mathcal{C}}(X) \cap V_{\mathcal{C}}(Y)$?
(Known to be true when $\mathcal{C}$ is a category of modules of a finite dimensional cocommutative Hopf algebra, and in some other cases or special types of modules, unknown in general.)