

# Gerstenhaber brackets on Hochschild cohomology of general twisted tensor products

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## Abstract

We present techniques for computing Gerstenhaber brackets on Hochschild cohomology of general twisted tensor product algebras. These techniques involve twisted tensor product resolutions and are based on recent results on Gerstenhaber brackets expressed on arbitrary bimodule resolutions.

## 1 Introduction

Čap, Schichl, and Vanžura [2] observed that whenever an algebra over a field has underlying vector space given by a tensor product of two subalgebras, it takes the form of a twisted tensor product algebra. Multiplication is given by a twisting map that determines how to move elements of one subalgebra past the other. In the special case that the twisting is given by a bicharacter, techniques were developed in [4] for computing the Lie algebra structure on the Hochschild cohomology of a twisted tensor product algebra. Here we generalize these techniques, describing how to compute Gerstenhaber brackets for a general twisted tensor product algebra on a twisted tensor product resolution from [13] based on the computational methods of [10]. We illustrate this with the Jordan plane, an example that was computed by Lopes and Solotar [9] using completely different methods. A special case of a twisted tensor product algebra is a skew group algebra, and there is a parallel development of Gerstenhaber bracket techniques for skew group algebras in [14]. We generalize some of those results here, pointing out additional necessary conditions in the general case. Gerstenhaber brackets are notoriously difficult to compute, and having a variety of techniques at hand is important. Our results here add to the collection of techniques available.

The contents of the paper are as follows: We define general twisted tensor product algebras  $A \otimes_{\tau} B$  in Section 2 and describe the construction of twisted tensor product

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resolutions of  $A \otimes_\tau B$ -bimodules [13]. We then recall from [5, 13] the case of bar and Koszul resolutions. Our main results are in Section 3 where we prove that under some conditions, Gerstenhaber brackets can be computed on twisted tensor product resolutions using techniques from [10]; see Theorem 3.11, which generalizes results in [4, 14]. We illustrate this with the Jordan plane in Section 4.

## 2 Twisted tensor product algebras and resolutions

In this section, we recall the notions of general twisted tensor product algebras from [2] and their twisted tensor product resolutions from [13]. More details may be found in those papers.

Let  $k$  be a field and denote  $\otimes := \otimes_k$ . Let  $A, B$  be  $k$ -algebras and let  $A^e, B^e$  be their respective enveloping algebras ( $A^e = A \otimes A^{op}$ ,  $B^e = B \otimes B^{op}$ ). A bijective  $k$ -linear map  $\tau : B \otimes A \rightarrow A \otimes B$  is called a *twisting map* if  $\tau(1_B \otimes a) = a \otimes 1_B$  and  $\tau(b \otimes 1_A) = 1_A \otimes b$  for all  $a \in A, b \in B$  where  $1_A, 1_B$  are the multiplicative identities of  $A, B$ , respectively, and the following diagram commutes:

$$\begin{array}{ccccc}
 B \otimes B \otimes A \otimes A & \xrightarrow{m_B \otimes m_A} & B \otimes A & \xrightarrow{\tau} & A \otimes B \\
 \downarrow 1 \otimes \tau \otimes 1 & & & & \uparrow m_A \otimes m_B \\
 B \otimes A \otimes B \otimes A & \xrightarrow{\tau \otimes \tau} & A \otimes B \otimes A \otimes B & \xrightarrow{1 \otimes \tau \otimes 1} & A \otimes A \otimes B \otimes B
 \end{array} \tag{2.1}$$

The *twisted tensor product algebra*, denoted  $A \otimes_\tau B$ , is the vector space  $A \otimes B$  with multiplication:

$$m_\tau : (A \otimes B) \otimes (A \otimes B) \xrightarrow{1 \otimes \tau \otimes 1} A \otimes A \otimes B \otimes B \xrightarrow{m_A \otimes m_B} A \otimes B.$$

This multiplication is associative provided  $\tau$  is a twisting map. The inverse map  $\tau^{-1} : A \otimes B \rightarrow B \otimes A$  satisfies analogous conditions, and there is an isomorphism of algebras  $A \otimes_\tau B \xrightarrow{\sim} B \otimes_{\tau^{-1}} A$  given by  $\tau^{-1}$ .

We will often work with algebras graded by the natural numbers: Suppose that  $A = \bigoplus_{n \in \mathbb{N}} A_n$  and  $B = \bigoplus_{n \in \mathbb{N}} B_n$ , where  $\mathbb{N}$  is understood to contain 0 and  $1_A \in A_0, 1_B \in B_0$ . Then the vector space  $A \otimes B$  is graded with  $(A \otimes B)_n = \bigoplus_{i+j=n} A_i \otimes B_j$  for all  $n$ , and  $B \otimes A$  is graded similarly. We say that a twisting map  $\tau$  is *graded* if  $\tau((B \otimes A)_n) = (A \otimes B)_n$  for all  $n$ , and that it is *strongly graded* if  $\tau(B_j \otimes A_i) = A_i \otimes B_j$  for all  $i, j$ .

An  $A$ -bimodule  $M$ , whose bimodule structure is given by  $\rho_A : A \otimes M \otimes A \rightarrow M$ , is said to be *compatible with  $\tau$*  if there exists a bijective  $k$ -linear map  $\tau_{B,M} : B \otimes M \rightarrow$

$M \otimes B$  such that the following diagram commutes:

$$\begin{array}{ccc}
B \otimes B \otimes M & \xrightarrow{1 \otimes \tau_{B,M}} & B \otimes M \otimes B & \xrightarrow{\tau_{B,M} \otimes 1} & M \otimes B \otimes B \\
m_B \otimes 1 \downarrow & & & & \downarrow 1 \otimes m_B \\
B \otimes M & \xrightarrow{\tau_{B,M}} & M \otimes B & & \\
1 \otimes \rho_A \uparrow & & & & \uparrow \rho_A \otimes 1 \\
B \otimes A \otimes M \otimes A & & A \otimes M \otimes A \otimes B & & \\
\tau \otimes 1 \otimes 1 \downarrow & & & & \uparrow 1 \otimes 1 \otimes \tau \\
A \otimes B \otimes M \otimes A & \xrightarrow{1 \otimes \tau_{B,M} \otimes 1} & A \otimes M \otimes B \otimes A & & 
\end{array}$$

We can analogously define how a  $B$ -bimodule  $N$  is compatible with  $\tau$  via a bijective  $k$ -linear map  $\tau_{N,A} : N \otimes A \rightarrow A \otimes N$ .

Note that the  $A^e$ -module  $M = A$  is itself compatible with  $\tau$ , taking  $\tau_{B,A}$  to be simply the map  $\tau$ : We can see that the upper part of the diagram above commutes by considering diagram (2.1) applied to elements for which the rightmost tensor factor is  $1_A$ . The bottom part commutes by iterating the commutativity expressed by diagram (2.1) and taking the leftmost tensor factor to be  $1_B$ .

We next define an  $A \otimes_\tau B$ -bimodule structure on the tensor product of compatible bimodules.

**Definition 2.2. Bimodule Structure :** Let  $M$  and  $N$  be  $A$ - and  $B$ -bimodules via  $\rho_A : A \otimes M \otimes A \rightarrow M$  and  $\rho_B : B \otimes N \otimes B \rightarrow N$ , respectively. Assume that  $M, N$  are compatible with  $\tau$  via  $\tau_{B,M}, \tau_{N,A}$ . Then  $M \otimes N$  has a natural structure of an  $A \otimes_\tau B$ -bimodule via  $\rho_{A \otimes_\tau B}$ , given by the following commutative diagram:

$$\begin{array}{ccc}
(A \otimes_\tau B) \otimes (M \otimes N) \otimes (A \otimes_\tau B) & \xrightarrow{\rho_{A \otimes_\tau B}} & M \otimes N \\
1 \otimes \tau_{B,M} \otimes \tau_{N,A} \otimes 1 \downarrow & & \uparrow \rho_A \otimes \rho_B \\
A \otimes M \otimes B \otimes A \otimes N \otimes B & \xrightarrow{1 \otimes 1 \otimes \tau \otimes 1 \otimes 1} & A \otimes M \otimes A \otimes B \otimes N \otimes B
\end{array}$$

Next we recall the notion of compatibility of resolutions. Let  $P_\bullet(M)$  be an  $A^e$ -projective resolution of  $M$  and  $P_\bullet(N)$  a  $B^e$ -projective resolution of  $N$ :

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & P_2(M) & \longrightarrow & P_1(M) & \longrightarrow & P_0(M) \longrightarrow M \longrightarrow 0, \\
\cdots & \longrightarrow & P_2(N) & \longrightarrow & P_1(N) & \longrightarrow & P_0(N) \longrightarrow N \longrightarrow 0.
\end{array}$$

We consider the complexes  $P_\bullet(N) \otimes A, A \otimes P_\bullet(N), P_\bullet(M) \otimes B, B \otimes P_\bullet(M)$ . As exact sequences of vector spaces, note that any  $k$ -linear maps  $\tau_{N,A} : N \otimes A \rightarrow A \otimes N$  and  $\tau_{B,M} : B \otimes M \rightarrow M \otimes B$  can be lifted to  $k$ -linear chain maps:

$$\tau_{P_\bullet(N),A} : P_\bullet(N) \otimes A \longrightarrow A \otimes P_\bullet(N), \quad \tau_{B,P_\bullet(M)} : B \otimes P_\bullet(M) \longrightarrow P_\bullet(M) \otimes B,$$

which we will denote more simply by  $\tau_{i,A} := \tau_{P_i(N),A}$  and  $\tau_{B,i} := \tau_{B,P_i(M)}$ . Given  $M$  an  $A$ -bimodule that is compatible with  $\tau$ , we say that a projective  $A^e$ -resolution  $P_\bullet(M)$  is

compatible with  $\tau$  if each  $P_i(M)$  is compatible with  $\tau$  via a map  $\tau_{B,i} : B \otimes P_i(M) \rightarrow P_i(M) \otimes B$  such that  $\tau_{B,\bullet}$  is a chain map lifting  $\tau_{B,M}$ . Given a  $B$ -bimodule  $N$  compatible with  $\tau$  we can analogously define how a projective  $B^e$ -resolution  $P_\bullet(N)$  is compatible with  $\tau$  via  $\tau_{\bullet,A}$ .

Provided the resolutions  $P_\bullet(M), P_\bullet(N)$  are compatible with  $\tau$ , by [13, Lemma 3.5], the total complex of the tensor product complex  $P_\bullet(M) \otimes P_\bullet(N)$  has homology concentrated in degree 0, where it is  $M \otimes N$ . If in addition, each  $(A \otimes_\tau B)^e$ -module  $P_i(M) \otimes P_j(N)$  is projective, it follows immediately that this complex is a projective resolution, called a *twisted tensor product resolution*:

**Theorem 2.3.** *Let  $M, N$  be  $A$ - and  $B$ -bimodules that are compatible with a twisting map  $\tau : B \otimes A \rightarrow A \otimes B$ . Let  $P_\bullet(M), P_\bullet(N)$  be projective  $A^e$ - and  $B^e$ -module resolutions of  $M$  and  $N$ , respectively, that are compatible with  $\tau$ . If each  $P_i(M) \otimes P_j(N)$  is a projective  $(A \otimes_\tau B)^e$ -module under the module structure given in Definition 2.2, then the total complex of  $P_\bullet(M) \otimes P_\bullet(N)$  is a projective  $(A \otimes_\tau B)^e$ -module resolution of  $M \otimes N$ .*

We will see directly that the hypotheses of the theorem are true of all the resolutions that we consider in this paper. Alternatively, see [13, Theorem 3.10] for additional hypotheses ensuring that all modules  $P_i(M) \otimes P_j(N)$  are projective. Next, we explain in detail the special cases where  $M = A, N = B$ , and  $P_\bullet(A), P_\bullet(B)$  are either bar resolutions or Koszul resolutions (when  $A, B$  are Koszul algebras). We will then be able to use these twisted tensor product resolutions to compute the Hochschild cohomology  $\mathrm{HH}^*(A \otimes_\tau B)$  as  $\mathrm{Ext}_{(A \otimes_\tau B)^e}^*(A \otimes_\tau B, A \otimes_\tau B)$ .

Consider the sequence of left  $A^e$ -modules (equivalently  $A$ -bimodules):

$$\dots \xrightarrow{d_3} A^{\otimes 4} \xrightarrow{d_2} A^{\otimes 3} \xrightarrow{d_1} A^{\otimes 2} \xrightarrow{m_A} A \rightarrow 0,$$

with differentials, for all  $n \geq 1$ :

$$d_n(a_0 \otimes \dots \otimes a_{n+1}) = \sum_{i=0}^n (-1)^i a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_{n+1}. \quad (2.4)$$

The *bar resolution*  $\mathbb{B}(A)$  is the truncated complex, that is,  $\mathbb{B}_n(A) = A^{\otimes(n+2)}$  for all  $n \geq 0$ . Since  $k$  is a field,  $\mathbb{B}(A)$  is a free left  $A^e$ -module resolution of  $A$ . The module structure can be expressed as  $\rho_{A,A^{\otimes n}} : A \otimes A^{\otimes n} \otimes A \rightarrow A^{\otimes n}$  where  $\rho_{A,A^{\otimes n}}(a \otimes c_1 \otimes \dots \otimes c_n \otimes b) = ac_1 \otimes \dots \otimes c_n b$  for all  $a, b, c_1, \dots, c_n$ , that is  $\rho_{A,A} = m_A(1 \otimes m_A)$  and  $\rho_{A,A^{\otimes n}} = m_A \otimes 1_{n-2} \otimes m_A$  for  $n \geq 2$ , where  $1_{n-2}$  denotes the identity map on  $A^{\otimes(n-2)}$ .

The bar resolution of  $A$  is compatible with  $\tau$  (see, e.g., [5] or [13, Proposition 2.20(i)]):

**Theorem 2.5.** *Let  $A, B$  be  $k$ -algebras and let  $\tau : B \otimes A \rightarrow A \otimes B$  be a twisting map. The bar resolutions  $\mathbb{B}(A)$  and  $\mathbb{B}(B)$  are compatible with  $\tau$ .*

The proof, as described in [13], is iteration of the commutativity of diagram (2.1).

Now let  $V$  be a finite dimensional vector space over  $k$  and let  $T(V) = \bigoplus_{n \in \mathbb{N}} T^n(V)$  denote the tensor algebra of  $V$ , so that  $T^n(V) = V \otimes \dots \otimes V$  ( $n$  tensor factors) for each

$n \in \mathbb{N}$ . Then  $T(V)$  may be considered to be a graded algebra, where for any  $v \in V$  we use  $|v|$  to denote its degree and assign  $|v| = 1$ .

Given a vector subspace  $R \subseteq V \otimes V = T^2(V)$ , set  $A = T(V)/(R)$  where  $(R)$  denotes the ideal generated by  $R$  in  $T(V)$ . Then  $A$  is a graded algebra generated by elements in degree 1 with relations  $R$  in degree 2. Consider the sequence of  $A$ -bimodules:

$$\cdots \xrightarrow{d_4} \mathbb{K}_3(A) \xrightarrow{d_3} A \otimes R \otimes A \xrightarrow{d_2} A \otimes V \otimes A \xrightarrow{d_1} A \otimes A \xrightarrow{m_A} A \longrightarrow 0,$$

where  $\mathbb{K}_0(A) = A \otimes A$ ,  $\mathbb{K}_1(A) = A \otimes V \otimes A$ ,  $\mathbb{K}_2(A) = A \otimes R \otimes A$  and

$$\mathbb{K}_n(A) = A \otimes \bigcap_{i=0}^{n-2} \left( V^{\otimes i} \otimes R \otimes V^{\otimes(n-i-2)} \right) \otimes A$$

for  $n \geq 3$ , with differentials given by (2.4) (that is, the same differentials as in the bar resolution, viewing each  $\mathbb{K}_n(A)$  as a subset of  $A^{\otimes(n+2)}$ ). Since  $m_A(R) = 0$ , the image of  $\mathbb{K}_n(A)$  under  $d_n$  is indeed contained in  $\mathbb{K}_{n-1}(A)$ . We see in this way that  $\mathbb{K}(A)$  is a subcomplex of the bar resolution.

We say that  $A$  is a *Koszul algebra* whenever the truncated complex  $\mathbb{K}(A)$  is a resolution of  $A$  as an  $A^e$ -module. This definition of Koszul algebra is equivalent to other definitions in the literature (see [7, Proposition 19] or [17, Theorem 3.4.6]). In this case, we say that  $\mathbb{K}(A)$  is the *Koszul resolution* of  $A$ .

Koszul resolutions are compatible with any strongly graded twisting map  $\tau$  [5, 13]. This follows for example by using techniques in the proof of [18, Proposition 1.8] (see also [6, Corollary 4.19] or [12, p. 90, Example 3]). The idea is that the above compatibility statements for the bar resolution descend to the Koszul resolution. Specifically, we have the following statement, which is essentially [13, Proposition 2.20(iii)]. We give just a sketch of a proof here, for completeness.

**Theorem 2.6.** *Let  $A$  be a Koszul algebra, let  $B$  be a graded algebra, and let  $\tau : B \otimes A \rightarrow A \otimes B$  be a strongly graded twisting map. Then the Koszul resolution  $\mathbb{K}(A)$  is compatible with  $\tau$ . If  $B$  is also a Koszul algebra, its Koszul resolution  $\mathbb{K}(B)$  is compatible with  $\tau$ .*

*Proof.* Define a map  $\tau_n : B \otimes A^{\otimes(n+2)} \rightarrow A^{\otimes(n+2)} \otimes B$  by iterating  $\tau$ : First let  $\tau_0 = (1 \otimes \tau)(\tau \otimes 1)$ , then let  $\tau_1 = (1 \otimes 1 \otimes \tau)(1 \otimes \tau \otimes 1)(\tau \otimes 1 \otimes 1)$ , and so on. We claim that the subcomplex  $\mathbb{K}(A)$  of the bar resolution is preserved by  $\tau_n$  in the following sense. As  $\tau$  is strongly graded,  $\tau_1(B \otimes V) \subseteq V \otimes B$ . Since  $\tau$  is a twisting map and  $m_A(R) = 0$ , we find that  $\tau_2(B \otimes R) \subseteq R \otimes B$ . In general,

$$\tau_n(B \otimes \mathbb{K}_n(A)) \subseteq \mathbb{K}_n(A) \otimes B$$

for all  $n$ . Note that the map  $\tau_n$  is a chain map since  $\tau$  is a twisting map and the differential is given by (2.4). The first statement now follows from Theorem 2.5, by restricting  $\tau_n$  to  $\mathbb{K}(A)$ . The second statement is similar.  $\square$

### 3 Gerstenhaber bracket for twisted tensor products

In this section, we begin by summarizing techniques from [10] for computing Gerstenhaber brackets on the Hochschild cohomology ring  $\mathrm{HH}^*(R)$  of a  $k$ -algebra  $R$ , as reformulated in [17, Section 6.4]. We will then take  $R$  to be a twisted tensor product algebra  $A \otimes_\tau B$  and develop further techniques for handling a twisted tensor product algebra specifically, generalizing work in [4, 14].

The graded Lie algebra structure on the Hochschild cohomology ring  $\mathrm{HH}^*(R)$  was historically defined on the bar resolution, with equivalent recent definitions on other resolutions such as in [10, 19]. Here we will simply take a formula from [10], stated in Theorem 3.3 below, to be our definition of the Gerstenhaber bracket, and refer to the cited literature for proof that it is equivalent to the historical definition.

Let  $K$  be a projective resolution of  $R$  as an  $R^e$ -module, with differential  $d$  and augmentation map  $\mu : K_0 \rightarrow R$ . Assume that  $(K, d)$  is a *counital differential graded coalgebra*, i.e. there is a diagonal map  $\Delta : K \rightarrow K \otimes_R K$  such that  $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$  and  $(\mu \otimes 1)\Delta = 1 = (1 \otimes \mu)\Delta$ .

Here and elsewhere, viewing  $K$  as a graded vector space, we adopt the following standard sign convention for tensor products of maps: If  $V, W, V', W'$  are graded vector spaces and  $g : V \rightarrow V', h : W \rightarrow W'$  are graded  $k$ -linear maps, define

$$(g \otimes h)(v \otimes w) = (-1)^{|h||v|} g(v) \otimes h(w)$$

for all homogeneous  $v \in V, w \in W$ , where  $|h|, |v|$  denote the degrees of  $h, v$ , respectively.

It can be shown (see, e.g., [17, Section 6.4]) that there exists an  $R$ -bimodule map  $\phi : K \otimes_R K \rightarrow K[1]$  for which

$$d\phi + \phi(d \otimes 1 + 1 \otimes d) = \mu \otimes 1 - 1 \otimes \mu, \quad (3.1)$$

where we take  $\mu$  to be the zero map on  $K_i$  for all  $i > 0$  and  $K[1]_i = K_{i+1}$  for all  $i$  so that  $\phi$  is a map of degree 1. The map  $\phi$  is called a *contracting homotopy* for  $\mu \otimes 1 - 1 \otimes \mu$ .

Let  $f \in \mathrm{Hom}_{R^e}(K_m, R)$  be a cocycle, that is  $fd_{m+1} = 0$ . We can view it as a map  $f : K \rightarrow R$  by setting  $f|_{K_n} = 0$  whenever  $n \neq m$ . Define  $\psi_f : K \rightarrow K[1 - m]$  by

$$\psi_f = \phi(1 \otimes f \otimes 1)\Delta^{(2)}, \quad (3.2)$$

where  $\Delta^{(2)} := (\Delta \otimes 1)\Delta$  and  $\phi$  is a contracting homotopy for  $\mu \otimes 1 - 1 \otimes \mu$  as defined above. In the expression (3.2),  $1 \otimes f \otimes 1$  is a map from  $K \otimes_R K \otimes_R K$  to  $K \otimes_R R \otimes_R K \cong K \otimes_R K$ , and we have identified these latter two isomorphic vector spaces. Note that  $\psi_f$  is a map of degree  $1 - m$  on  $K$  by definition.

The following theorem is from [10] with slightly different hypotheses, and is stated in this form as [17, Theorem 6.4.5]. We will take the formula in the theorem for the Gerstenhaber bracket as a definition for our purposes in the rest of this paper, since our resolutions will always be counital differential graded coalgebras.

**Theorem 3.3.** *Let  $K$  be a projective resolution of  $R$  as an  $R^e$ -module that is a counital differential graded coalgebra. Let  $f \in \text{Hom}_{R^e}(K_m, R)$  and  $g \in \text{Hom}_{R^e}(K_n, R)$  be cocycles. Define  $\psi_f$  as in (3.2), and similarly  $\psi_g$ . Then*

$$[f, g] := f\psi_g - (-1)^{(m-1)(n-1)}g\psi_f, \quad (3.4)$$

as a function in  $\text{Hom}_{R^e}(K_{m+n-1}, R)$ , represents the Gerstenhaber bracket on Hochschild cohomology.

We next explain how to obtain a map  $\phi$  satisfying equation (3.1) when  $K$  is a twisted tensor product resolution. We may then use  $\phi$  in the formula (3.2) in order to compute Gerstenhaber brackets via formula (3.4).

Let  $A$  and  $B$  be  $k$ -algebras and let  $\tau : B \otimes A \rightarrow A \otimes B$  be a twisting map. Let

$$\begin{aligned} P : \quad & \cdots \xrightarrow{d_2^P} P_1 \xrightarrow{d_1^P} P_0 \xrightarrow{\mu_P} A \rightarrow 0, \\ Q : \quad & \cdots \xrightarrow{d_2^Q} Q_1 \xrightarrow{d_1^Q} Q_0 \xrightarrow{\mu_Q} B \rightarrow 0 \end{aligned}$$

be an  $A^e$ -projective resolution of  $A$  and a  $B^e$ -projective resolution of  $B$ , respectively. By Theorem 2.3, if  $P$  and  $Q$  are compatible with  $\tau$  and  $P_i \otimes Q_j$  is a projective  $(A \otimes_\tau B)^e$ -module for all  $i, j$ , then the total complex of  $P \otimes Q$  is an  $(A \otimes_\tau B)^e$ -projective resolution of  $A \otimes_\tau B$ . We will denote this resolution by  $P \otimes_\tau Q$ . In particular, this is the case when  $P$  and  $Q$  are both bar resolutions or they are both Koszul resolutions and  $\tau$  is strongly graded, by Theorems 2.5 and 2.6.

We will be interested in resolutions  $P, Q$  for which there exists an  $A \otimes_\tau B$ -bimodule chain map

$$\sigma : (P \otimes_\tau Q) \otimes_{A \otimes_\tau B} (P \otimes_\tau Q) \rightarrow (P \otimes_A P) \otimes_\tau (Q \otimes_B Q) \quad (3.5)$$

such that

$$\mu_P \otimes \mu_Q \otimes 1_P \otimes 1_Q - 1_P \otimes 1_Q \otimes \mu_P \otimes \mu_Q = (\mu_P \otimes 1_P \otimes \mu_Q \otimes 1_Q - 1_P \otimes \mu_P \otimes 1_Q \otimes \mu_Q) \sigma \quad (3.6)$$

as a map from  $(P \otimes_\tau Q) \otimes_{A \otimes_\tau B} (P \otimes_\tau Q)$  to  $P \otimes_\tau Q$ . We will see that such a map  $\sigma$  exists for bar and Koszul resolutions in particular in Lemma 3.8 below.

First we define a map that will be a forerunner to  $\sigma$  for these resolutions: Assume for now that  $P$  and  $Q$  are both bar resolutions. Let  $\sigma' : P \otimes Q \otimes P \otimes Q \rightarrow P \otimes P \otimes Q \otimes Q$  be the map given in each degree as  $\sigma'_{r,s,t,u} : P_r \otimes Q_s \otimes P_t \otimes Q_u \rightarrow P_r \otimes P_t \otimes Q_s \otimes Q_u$  for  $r, s, t, u \in \mathbb{N}$  where

$$\begin{aligned} \sigma'_{r,s,t,u} = & (-1)^{st} (1_{r+2} \otimes \tau_{B,t} \otimes 1_{s+1} \otimes 1_{u+2}) \circ (1_{r+2} \otimes 1 \otimes \tau_{B,t} \otimes 1_s \otimes 1_{u+2}) \\ & \circ \cdots \circ (1_{r+2} \otimes 1_s \otimes \tau_{B,t} \otimes 1 \otimes 1_{u+2}) \circ (1_{r+2} \otimes 1_{s+1} \otimes \tau_{B,t} \otimes 1_{u+2}). \end{aligned} \quad (3.7)$$

This definition says that  $\sigma'$  is the map that takes the rightmost element in the tensor product  $Q_s$ , passes it through  $P_t$  via  $\tau_{B,t}$ , then takes the second rightmost element and proceeds likewise, and so on, till we have passed factors in  $Q_s$  to the right side of factors in  $P_t$ . However, we could proceed in a symmetric way by first taking the leftmost element

in the tensor product of  $P_t$ , passing it through  $Q_s$  via  $\tau_{s,A}$ , then proceeding analogously as before, till we have passed all elements forming  $P_t$  to the left side of  $Q_s$ . By properties of the twisting map  $\tau$ , these two constructions will be the same.

Next assume that  $P$  and  $Q$  are Koszul resolutions and  $\tau$  is strongly graded. An argument similar to the proof of Theorem 2.6 shows that  $\sigma'$ , defined similarly by equation (3.7) in each degree, is indeed a well-defined map on Koszul resolutions.

**Lemma 3.8.** *Let  $P, Q$  be bar resolutions or Koszul resolutions in case  $\tau$  is strongly graded and define  $\sigma'$  as in equation (3.7). Then  $\sigma'$  induces a chain map  $\sigma$  as in (3.5) that is an isomorphism of  $(A \otimes_\tau B)^e$ -modules in each degree, lifting the canonical isomorphism*

$$(A \otimes_\tau B) \otimes_{A \otimes_\tau B} (A \otimes_\tau B) \xrightarrow{\sim} A \otimes_\tau B.$$

*In particular, condition (3.6) holds.*

*Proof.* First we note that  $\sigma'$ , defined by equation (3.7) in each degree, induces a map

$$\sigma : (P \otimes_\tau Q) \otimes_{A \otimes_\tau B} (P \otimes_\tau Q) \rightarrow (P \otimes_A P) \otimes_\tau (Q \otimes_B Q)$$

as claimed: Similarly to the proof of [14, Lemma 4.1], the map given by the composition

$$(P \otimes Q) \otimes (P \otimes Q) \xrightarrow{1 \otimes \tau \otimes 1} P \otimes P \otimes Q \otimes Q \longrightarrow (P \otimes_A P) \otimes (Q \otimes_B Q),$$

where the latter map is the canonical surjection, is  $A \otimes_\tau B$ -middle linear. Therefore it induces a well-defined map  $\sigma$  as claimed. Further,  $\sigma$  is a chain map since  $\tau$  is compatible with the multiplication maps  $\mu_A, \mu_B$ , the bimodule actions are defined in terms of multiplication, and Koszul resolutions are subcomplexes of bar resolutions. Finally,  $\sigma$  is a bijection; an inverse map can be defined similarly using the inverse of the twisting map  $\tau$ .  $\square$

Other settings where a map  $\sigma$  exists that satisfies condition (3.6) are skew group algebras [14, Remark 4.4] and the Jordan plane of Section 4 below. We see next that condition (3.6) allows us to define a contracting homotopy, generalizing the cases where the twisting is given by a bicharacter [4, Lemma 3.5] or by a group action resulting in a skew group algebra [14, Theorem 4.6]. Our proof is essentially the same as in these two cases; we provide details for completeness.

**Lemma 3.9.** *Let  $P, Q$  be projective  $A^e$ - and  $B^e$ -module resolutions, respectively, both compatible with  $\tau$ , for which  $P \otimes_\tau Q$  is a projective  $(A \otimes_\tau B)^e$ -module resolution of  $A \otimes_\tau B$ . Let  $\phi_P$  and  $\phi_Q$  be contracting homotopies for  $\mu_P \otimes 1_P - 1_P \otimes \mu_P$  and  $\mu_Q \otimes 1_Q - 1_Q \otimes \mu_Q$ , respectively. Assume there is a chain map  $\sigma$  as in (3.5) satisfying condition (3.6). Define  $\phi = \phi_{P \otimes_\tau Q} : (P \otimes_\tau Q) \otimes_{A \otimes_\tau B} (P \otimes_\tau Q) \rightarrow (P \otimes_\tau Q)$  by*

$$\phi := (\phi_P \otimes \mu_Q \otimes 1_Q + 1_P \otimes \mu_P \otimes \phi_Q)\sigma. \quad (3.10)$$

*Then  $\phi$  is a contracting homotopy for  $\mu_P \otimes \mu_Q \otimes 1_P \otimes 1_Q - 1_P \otimes 1_Q \otimes \mu_P \otimes \mu_Q$ .*

*Proof.* To shorten notation, as in [10], we define chain maps  $F_P^l, F_P^r, F_P : P \otimes_A P \rightarrow P$  by

$$F_P^l = \mu_P \otimes 1_P, \quad F_P^r = 1_P \otimes \mu_P, \quad \text{and} \quad F_P = F_P^l - F_P^r,$$

where  $1_P$  is the identity map on  $P$ . We define  $F_Q^l, F_Q^r,$  and  $F_Q$  similarly. As  $F_P^l, F_Q^l, F_P^r, F_Q^r,$  and  $\sigma$  are chain maps, they commute with the differentials, so for example  $d_P F_P^r = F_P^r d_{P \otimes_A P}$ , and similar equations hold for the other maps. Therefore by the above formulas, notation, and the definition of the differential on a tensor product of complexes,

$$\begin{aligned} d\phi + \phi d &= (d \otimes 1 + 1 \otimes d)(\phi_P \otimes F_Q^l + F_P^r \otimes \phi_Q)\sigma \\ &\quad + (\phi_P \otimes F_Q^l + F_P^r \otimes \phi_Q)\sigma(d \otimes 1 + 1 \otimes d) \\ &= (d\phi_P \otimes F_Q^l + dF_P^r \otimes \phi_Q + \phi_P \otimes dF_Q^l + F_P^r \otimes d\phi_Q \\ &\quad + \phi_P d \otimes F_Q^l + \phi_P \otimes F_Q^l d + F_P^r d \otimes \phi_Q + F_P^r \otimes \phi_Q d)\sigma \\ &= ((d\phi_P + \phi_P d) \otimes F_Q^l + F_P^r \otimes (d\phi_Q + \phi_Q d))\sigma. \end{aligned}$$

Applying (3.1) to rewrite  $d\phi_P + \phi_P d$  and  $d\phi_Q + \phi_Q d$ , this expression is equal to

$$\begin{aligned} (F_P \otimes F_Q^l + F_P^r \otimes F_Q)\sigma &= ((F_P^l - F_P^r) \otimes F_Q^l + F_P^r \otimes (F_Q^l - F_Q^r))\sigma \\ &= (F_P^l \otimes F_Q^l - F_P^r \otimes F_Q^r)\sigma. \end{aligned}$$

By condition (3.6), this is equal to  $\mu_P \otimes \mu_Q \otimes 1_P \otimes 1_Q - 1_P \otimes 1_Q \otimes \mu_P \otimes \mu_Q$ , as desired.  $\square$

As a consequence of Lemma 3.9, we may now define a contracting homotopy  $\phi_{P \otimes_\tau Q}$  from knowledge of  $\phi_P, \phi_Q$  provided there is a chain map  $\sigma$  satisfying condition (3.6). We may then compute Gerstenhaber brackets from these maps. We have thus proven:

**Theorem 3.11.** *Let  $A$  and  $B$  be  $k$ -algebras and let  $P$  and  $Q$  be a projective  $A^e$ - and  $B^e$ -module resolution of  $A$  and  $B$ , respectively. Assume that  $P \otimes_\tau Q$  is a projective  $(A \otimes_\tau B)^e$ -module resolution of  $A \otimes_\tau B$  that is a counital differential graded coalgebra and that there is a chain map  $\sigma$  satisfying condition (3.6). Then Gerstenhaber brackets of Hochschild cocycles on  $P \otimes_\tau Q$  are given by formula (3.4) via formulas (3.2) and (3.10).*

By Lemma 3.8, the theorem applies whenever  $P, Q$  are bar resolutions, or when  $P, Q$  are Koszul resolutions and  $\tau$  is strongly graded. In the next section, we give an example of yet another setting in which a suitable chain map  $\sigma$  may be defined, and so the theorem applies. As already mentioned, the theorem applies more generally to twisted tensor products where the twisting is given by a bicharacter on grading groups [4, Section 3] and to skew group algebras [14, Theorem 4.9].

## 4 Jordan plane

In this section, we will illustrate the twisted tensor product techniques of Sections 2 and 3 with a small example. Let  $A = k[x]$  and  $B = k[y]$ . Let  $\tau : B \otimes A \rightarrow A \otimes B$  be defined by

$$\tau(y \otimes x) = x \otimes y + x^2 \otimes 1,$$

extended to  $B \otimes A$  by requiring  $\tau$  to be a twisting map. Its inverse is given by

$$\tau^{-1}(x \otimes y) = y \otimes x - 1 \otimes x^2.$$

There is an isomorphism

$$A \otimes_{\tau} B \cong k\langle x, y \rangle / (yx - xy - x^2).$$

Accordingly, we will write elements of the ring  $A \otimes_{\tau} B$  as (noncommutative) polynomials in indeterminates  $x$  and  $y$ , omitting the tensor product symbol in notation for these elements. We will identify the algebras  $A$  and  $B$  with subalgebras of  $k\langle x, y \rangle / (yx - xy - x^2)$  in this way.

We call  $A \otimes_{\tau} B$  the *Jordan plane*. Its Hochschild cohomology and in particular the Gerstenhaber algebra structure was computed by Lopes and Solotar [9] as one example in a larger class of algebras, using completely different techniques than ours: Their resolution was constructed using a method of Chouhy and Solotar [3], analogous to the Anick resolution, and their Gerstenhaber brackets were computed using the derivation operator method of Suárez-Álvarez [16] that applies specifically to brackets with Hochschild 1-cocycles (that is, derivations). See also [15, Theorem 4.6] for derivations on the Jordan plane and their Gerstenhaber brackets, and [1] for a generalization. Our approach is based on techniques from Section 3 and [10, 13]. We begin by explicitly constructing a twisted tensor product resolution  $P \otimes_{\tau} Q$  for  $A \otimes_{\tau} B$ .

Note that  $A$  and  $B$  are Koszul algebras and  $\tau$  is graded but not strongly graded. Thus Lemma 3.8 does not apply. Nonetheless we will construct a twisted tensor product resolution with a diagonal map  $\Delta$  and a map  $\sigma$  to which Theorem 3.11 applies. We will then be able to compute Gerstenhaber brackets on the Hochschild cohomology ring  $\mathrm{HH}^*(A \otimes_{\tau} B)$  as stated in Theorem 3.11.

Let  $P, Q$  be the Koszul resolutions for  $A, B$ , e.g.  $P$  is given by

$$P: \quad 0 \longrightarrow A \otimes A \xrightarrow{\tilde{d}_1 = (x \otimes 1 - 1 \otimes x)} A \otimes A \xrightarrow{m_A} A \longrightarrow 0,$$

where we have identified  $A \otimes V \otimes A$  with  $A \otimes A$  via canonical isomorphism ( $V$  is a one-dimensional vector space), and similarly  $Q$ . We use the notation for twisting maps as before on Koszul resolutions, under this identification. We will explicitly construct the twisted tensor product resolution  $P \otimes_{\tau} Q$  for  $A \otimes_{\tau} B$ .

We will choose some notation to help keep track of homological degrees of the free modules in the resolutions  $P, Q$ : For  $i \in \{0, 1\}$ , let  $e_i = 1 \otimes 1$ , a free generator of  $P_i$  as  $A^e$ -module. Similarly let  $e'_i = 1 \otimes 1$ , a free generator of  $Q_i$  as  $B^e$ -module. Calculations show that maps  $\tau_{0,A}$  and  $\tau_{1,A}$  may be given by

$$\begin{aligned} \tau_{0,A} &= (\tau \otimes 1_B)(1_B \otimes \tau), \\ \tau_{1,A}(e'_1 \otimes x) &= x \otimes e'_1. \end{aligned}$$

We also have

$$\tau_{1,A}^{-1}(x \otimes e'_1) = e'_1 \otimes x.$$

Since  $A$  is a free algebra on the generator  $x$ ,  $B \otimes B$  is a free  $B^e$ -module on the generator  $1 \otimes 1$ , and  $\tau$  is a twisting map, the above value of  $\tau_{1,A}$  on  $e'_1 \otimes x$  is sufficient to determine all other values, and similarly for  $\tau_{1,A}^{-1}$ . Maps  $\tau_{B,0}$  and  $\tau_{B,1}$  are given by

$$\begin{aligned}\tau_{B,0} &= (1_A \otimes \tau)(\tau \otimes 1_A), \\ \tau_{B,1}(y \otimes e_1) &= e_1 \otimes y + xe_1 \otimes 1 + e_1x \otimes 1.\end{aligned}$$

We also have

$$\tau_{B,1}^{-1}(e_1 \otimes y) = y \otimes e_1 - 1 \otimes xe_1 - 1 \otimes e_1x.$$

The total complex  $P_\bullet \otimes_\tau Q_\bullet$  is

$$0 \longrightarrow P_1 \otimes Q_1 \xrightarrow{d_2} (P_1 \otimes Q_0) \oplus (P_0 \otimes Q_1) \xrightarrow{d_1} P_0 \otimes Q_0 \longrightarrow 0 \quad (4.1)$$

with differentials as follows. Note first that each  $(A \otimes_\tau B)^e$ -module  $P_i \otimes Q_j$  is a free  $(A \otimes_\tau B)^e$ -module of rank one: A calculation shows that

$$P_i \otimes Q_j := A \otimes A \otimes B \otimes B \xrightarrow{1 \otimes \tau^{-1} \otimes 1} (A \otimes_\tau B) \otimes (A \otimes_\tau B)^{op}$$

is an isomorphism of  $(A \otimes_\tau B)^e$ -modules. We apply the standard formula for the differential on a tensor product of complexes as well as the maps  $\tau_{i,A}^{-1}$ ,  $\tau_{B,i}^{-1}$  given above in order to express elements in the resolution in terms of the free basis elements  $e_0 \otimes e'_0$ ,  $e_0 \otimes e'_1$ ,  $e_1 \otimes e'_0$ ,  $e_1 \otimes e'_1$ . We calculate as follows (recall that we write elements of the ring  $A \otimes_\tau B$  as noncommutative polynomials in  $x, y$ , omitting the tensor symbol in notation for these elements):

$$\begin{aligned}d_1(e_1 \otimes e'_0) &= \tilde{d}_1(e_1) \otimes e'_0 \\ &= xe_0 \otimes e'_0 - e_0x \otimes e'_0 = xe_0 \otimes e'_0 - e_0 \otimes e'_0x \\ &= (x \otimes 1 - 1 \otimes x)(e_0 \otimes e'_0), \\ d_1(e_0 \otimes e'_1) &= e_0 \otimes \tilde{d}_1(e'_1) \\ &= e_0 \otimes ye'_0 - e_0 \otimes e'_0y = ye_0 \otimes e'_0 - e_0 \otimes e'_0y \\ &= (y \otimes 1 - 1 \otimes y)(e_0 \otimes e'_0), \\ d_2(e_1 \otimes e'_1) &= \tilde{d}_1(e_1) \otimes e'_1 - e_1 \otimes \tilde{d}_1(e'_1) \\ &= xe_0 \otimes e'_1 - e_0x \otimes e'_1 - e_1 \otimes ye'_0 + e_1 \otimes e'_0y \\ &= xe_0 \otimes e'_1 - e_0 \otimes e'_1x - ye_1 \otimes e'_0 + xe_1 \otimes e'_0 + e_1x \otimes e'_0 + e_1 \otimes e'_0y \\ &= xe_0 \otimes e'_1 - e_0 \otimes e'_1x - ye_1 \otimes e'_0 + xe_1 \otimes e'_0 + e_1 \otimes e'_0x + e_1 \otimes e'_0y \\ &= (x \otimes 1 - 1 \otimes x)(e_0 \otimes e'_1) + (x \otimes 1 + 1 \otimes x - y \otimes 1 + 1 \otimes y)(e_1 \otimes e'_0).\end{aligned}$$

We next find expressions for Hochschild cocycles on which to use the techniques of Section 3: Apply  $\text{Hom}_{(A \otimes_\tau B)^e}(-, A \otimes_\tau B)$  to sequence (4.1). Since each  $P_i \otimes Q_j$  is isomorphic to the free  $(A \otimes_\tau B)^e$ -module  $(A \otimes_\tau B)^e$ , we find that

$$\text{Hom}_{(A \otimes_\tau B)^e}(P_i \otimes Q_j, A \otimes_\tau B) \cong A \otimes_\tau B$$

for each  $i, j$ , and thus the resulting complex becomes

$$0 \longrightarrow A \otimes_{\tau} B \xrightarrow{d_1^*} (A \otimes_{\tau} B) \oplus (A \otimes_{\tau} B) \xrightarrow{d_2^*} A \otimes_{\tau} B \longrightarrow 0. \quad (4.2)$$

By definition, we have

$$\mathrm{HH}^*(A \otimes_{\tau} B) = \mathrm{Ker}(d_1^*) \oplus \mathrm{Ker}(d_2^*)/\mathrm{Im}(d_1^*) \oplus (A \otimes_{\tau} B)/\mathrm{Im}(d_2^*). \quad (4.3)$$

From now on, we assume the characteristic of  $k$  is 0. The cohomology in positive characteristic can also be found from this complex, but it will be different.

By [15, Theorem 2.2], since  $\mathrm{char}(k) = 0$ , the center of  $A \otimes_{\tau} B$  is  $Z(A \otimes_{\tau} B) \cong k$ . Therefore  $\mathrm{HH}^0(A \otimes_{\tau} B) \cong k$ , which is precisely  $\mathrm{Ker}(d_1^*)$ .

Now we describe  $\mathrm{HH}^1(A \otimes_{\tau} B)$ . Recall that  $\mathrm{HH}^1(A \otimes_{\tau} B)$  is isomorphic to the space of derivations of  $A \otimes_{\tau} B$  modulo inner derivations, called outer derivations. The following theorem, which is written in a slightly different way and proven in [15, Theorem 4.6], directly provides outer derivations of  $A \otimes_{\tau} B$ . In our notation, viewing a derivation  $\partial$  as a Hochschild 1-cocycle on our resolution (4.1), it will take  $e_0 \otimes e'_1$  to  $\partial(y)$  and  $e_1 \otimes e'_0$  to  $\partial(x)$ . See also [1] or [9] for a more general setting.

**Theorem 4.4.** *If  $\mathrm{char}(k) = 0$ , then each derivation  $\partial$  of  $A \otimes_{\tau} B$  can be represented in the form  $\partial(y) = \alpha x + p + ad w(y)$ ,  $\partial(x) = p'x + ad w(x)$ , where  $\alpha \in k$ ,  $p \in k[y]$ ,  $p'$  is the derivative of  $p$  with respect to  $y$  in the usual sense,  $w \in A \otimes_{\tau} B$ , and  $ad w(\lambda) = w\lambda - \lambda w$  for  $\lambda \in A \otimes_{\tau} B$ .*

As a consequence of the theorem,  $\mathrm{HH}^1(A \otimes_{\tau} B) \cong k \oplus k[y]$ , which can also be shown directly from complex (4.2).

Lastly, we describe  $\mathrm{HH}^2(A \otimes_{\tau} B)$ . Calculations show that the image of  $d_2^*$  in  $A \otimes_{\tau} B$  is the ideal generated by  $x$ , so

$$\mathrm{HH}^2(A \otimes_{\tau} B) = (A \otimes_{\tau} B)/\mathrm{Im}(d_2^*) \cong k[y].$$

Finally, (4.3) becomes

$$\mathrm{HH}^*(A \otimes_{\tau} B) \cong k \oplus (k \oplus k[y]) \oplus k[y],$$

where the first copy of  $k$  is in homological degree 0, the middle two summands  $k \oplus k[y]$  are in degree 1, and the last copy of  $k[y]$  is in degree 2. (Cf. [9, Corollary 3.11].)

We will next find a diagonal map  $\Delta : P \otimes_{\tau} Q \rightarrow (P \otimes_{\tau} Q) \otimes_{A \otimes_{\tau} B} (P \otimes_{\tau} Q)$  for use in computing Gerstenhaber brackets on  $\mathrm{HH}^*(A \otimes_{\tau} B)$ . In order to do this, we will need to consider the total complex  $Q \cdot \otimes_{\tau^{-1}} P$  and the differentials  $\hat{d}_1, \hat{d}_2$ :

$Q \cdot \otimes_{\tau^{-1}} P :$

$$0 \longrightarrow Q_1 \otimes P_1 \xrightarrow{\hat{d}_2} (Q_1 \otimes P_0) \oplus (Q_0 \otimes P_1) \xrightarrow{\hat{d}_1} Q_0 \otimes P_0 \longrightarrow 0$$

(Recall that  $A \otimes_\tau B \cong B \otimes_{\tau^{-1}} A$  as algebras, and we define  $Q_\bullet \otimes_{\tau^{-1}} P_\bullet$  as a projective resolution of  $(B \otimes_{\tau^{-1}} A)^e$ -modules, equivalently of  $(A \otimes_\tau B)^e$ -modules, via the techniques of Section 2.) We find the differentials  $\hat{d}_2, \hat{d}_1$  in the same way as we found the differentials for  $P_\bullet \otimes_\tau Q_\bullet$ :

$$\begin{aligned}\hat{d}_1(e'_1 \otimes e_0) &= ye'_0 \otimes e_0 - e'_0 \otimes e_0y, \\ \hat{d}_1(e'_0 \otimes e_1) &= xe'_0 \otimes e_0 - e'_0 \otimes e_0x, \\ \hat{d}_2(e'_1 \otimes e_1) &= (1 \otimes x - x \otimes 1)(e'_1 \otimes e_0) + (y \otimes 1 - 1 \otimes y - x \otimes 1 - 1 \otimes x)(e'_0 \otimes e_1).\end{aligned}$$

Now, consider the complexes  $Q_\bullet \otimes_{\tau^{-1}} P_\bullet$  and  $P_\bullet \otimes_\tau Q_\bullet$ . We wish to define  $(A \otimes_\tau B)^e$ -module homomorphisms  $\tau_0, \tau_1$  and  $\tau_2$  so that  $\tau_\bullet$  is a chain map from  $Q_\bullet \otimes P_\bullet$  to  $P_\bullet \otimes Q_\bullet$ . Calculations show that the following values define such a chain map

$$\tau_0(e'_0 \otimes e_0) = e_0 \otimes e'_0, \quad \tau_1(e'_1 \otimes e_0) = e_0 \otimes e'_1, \quad \tau_1(e'_0 \otimes e_1) = e_1 \otimes e'_0, \quad \tau_2(e'_1 \otimes e_1) = -e_1 \otimes e'_1.$$

We may now set  $\sigma'_\bullet = 1_P \otimes \tau_\bullet \otimes 1_Q$  which induces a map

$$\sigma_\bullet : (P \otimes_\tau Q) \otimes_{A \otimes_\tau B} (P \otimes_\tau Q) \rightarrow (P \otimes_A P) \otimes_\tau (Q \otimes_B Q),$$

similar to the setting in Lemma 3.8. We may check that condition (3.6) holds.

A diagonal map  $\Delta_P$  on  $P$  is given by  $\Delta_P(e_0) = e_0 \otimes e_0$ ,  $\Delta_P(e_1) = e_0 \otimes e_1 + e_1 \otimes e_0$ , and similarly  $\Delta_Q$  on  $Q$ . (These are the maps given by viewing  $P, Q$  as subcomplexes of bar complexes.)

We define a diagonal map  $\Delta$  on  $P \otimes_\tau Q$ , taking  $\Delta$  to be the composition of the following maps in each degree:

$$P \otimes Q \xrightarrow{\Delta'_P \otimes \Delta'_Q} (P \otimes P) \otimes (Q \otimes Q) \xrightarrow{1_P \otimes \tau_\bullet^{-1} \otimes 1_Q} P \otimes Q \otimes P \otimes Q \rightarrow (P \otimes Q) \otimes_{A \otimes_\tau B} (P \otimes Q),$$

where the last map is a quotient map, and by  $\Delta'_P$  we mean the map to  $P \otimes P$  defined by an analogous formula as that of the diagonal map to  $P \otimes_A P$ , and similarly  $\Delta'_Q$ . By construction,  $\Delta$  is an  $(A \otimes_\tau B)^e$ -module homomorphism and a chain map. Under this composition of maps, we find that

$$\begin{aligned}\Delta(e_0 \otimes e'_0) &= (e_0 \otimes e'_0) \otimes (e_0 \otimes e'_0), \\ \Delta(e_1 \otimes e'_0) &= (e_0 \otimes e'_0) \otimes (e_1 \otimes e'_0) + (e_1 \otimes e'_0) \otimes (e_0 \otimes e'_0), \\ \Delta(e_0 \otimes e'_1) &= (e_0 \otimes e'_1) \otimes (e_0 \otimes e'_0) + (e_0 \otimes e'_0) \otimes (e_0 \otimes e'_1), \\ \Delta(e_1 \otimes e'_1) &= (e_0 \otimes e'_0) \otimes (e_1 \otimes e'_1) - (e_0 \otimes e'_1) \otimes (e_1 \otimes e'_0) \\ &\quad + (e_1 \otimes e'_0) \otimes (e_0 \otimes e'_1) + (e_1 \otimes e'_1) \otimes (e_0 \otimes e'_0).\end{aligned}$$

Direct calculations show that this diagonal map  $\Delta$  makes  $P \otimes_\tau Q$  a counital differential graded coalgebra, so the hypothesis of Theorem 3.3 holds. We may now apply Theorem 3.11 to calculate Gerstenhaber brackets. By formulas (3.2) and (3.10),  $\psi_f = \phi(1 \otimes f \otimes 1)\Delta^{(2)}$  and  $\phi = (\phi_P \otimes \mu_Q \otimes 1_Q + (-1)^{(i+p)} 1_P \otimes \mu_P \otimes \phi_Q)\sigma$ . We will use

formulas for  $\phi_P$  and  $\phi_Q$  from [10, Section 4]:  $\phi_P(e_0 \otimes x^t e_0) = \sum_{i=0}^{t-1} x^i e_1 x^{t-i-1}$  and similarly  $\phi_Q$ . These values of  $\phi$  and  $\Delta$  may be used to calculate the Gerstenhaber bracket of any two Hochschild cocycles via Theorem 3.3. We do one such calculation to explain the technique.

Let  $f \in \text{HH}^1(A \otimes_\tau B) \cong k \oplus k[y]$  and  $g \in \text{HH}^2(A \otimes_\tau B) \cong k[y]$ . We take  $f$  to correspond to  $y$  and  $g$  to  $y^3$ , so that

$$f(e_0 \otimes e'_1) = y, \quad f(e_1 \otimes e'_0) = x, \quad g(e_1 \otimes e'_1) = y^3.$$

We calculate the value of the bracket  $[f, g]$  on  $e_1 \otimes e'_1$ :

$$\begin{aligned} f\psi_g(e_1 \otimes e'_1) &= f\phi(1 \otimes g \otimes 1)(\Delta \otimes 1)(\Delta(e_1 \otimes e'_1)) \\ &= f\phi(1 \otimes g \otimes 1)(\Delta \otimes 1)((e_0 \otimes e'_0) \otimes (e_1 \otimes e'_1) - (e_0 \otimes e'_1) \otimes (e_1 \otimes e'_0) \\ &\quad + (e_1 \otimes e'_0) \otimes (e_0 \otimes e'_1) + (e_1 \otimes e'_1) \otimes (e_0 \otimes e'_0)) \\ &= f\phi((1 \otimes g \otimes 1)[(e_0 \otimes e'_0) \otimes (e_0 \otimes e'_0) \otimes (e_1 \otimes e'_1) \\ &\quad - (e_0 \otimes e'_1) \otimes (e_0 \otimes e'_0) \otimes (e_1 \otimes e'_0) - (e_0 \otimes e'_0) \otimes (e_0 \otimes e'_1) \otimes (e_1 \otimes e'_0) \\ &\quad + (e_0 \otimes e'_0) \otimes (e_1 \otimes e'_0) \otimes (e_0 \otimes e'_1) + (e_1 \otimes e'_0) \otimes (e_0 \otimes e'_0) \otimes (e_0 \otimes e'_1) \\ &\quad + (e_0 \otimes e'_0) \otimes (e_1 \otimes e'_1) \otimes (e_0 \otimes e'_0) - (e_0 \otimes e'_1) \otimes (e_1 \otimes e'_0) \otimes (e_0 \otimes e'_0) \\ &\quad + (e_1 \otimes e'_0) \otimes (e_0 \otimes e'_1) \otimes (e_0 \otimes e'_0) + (e_1 \otimes e'_1) \otimes (e_0 \otimes e'_0) \otimes (e_0 \otimes e'_0)]) \\ &= f\phi(e_0 \otimes e'_0 y^3 \otimes e_0 \otimes e'_0) \\ &= f((\phi_P \otimes \mu_Q \otimes 1_Q) + (1_P \otimes \mu_P \otimes \phi_Q))(e_0 \otimes e_0 \otimes e'_0 y^3 \otimes e'_0) \\ &= f(y^2 e_0 \otimes e'_1 + y e_0 \otimes e'_1 y + e_0 \otimes e'_1 y^2) \\ &= y^3 + y^3 + y^3 = 3y^3, \end{aligned}$$

and

$$\begin{aligned} g\psi_f(e_1 \otimes e'_1) &= g\phi(1 \otimes f \otimes 1)(\Delta \otimes 1)(\Delta(e_1 \otimes e'_1)) \\ &= g\phi(1 \otimes f \otimes 1)(\Delta \otimes 1)((e_0 \otimes e'_0) \otimes (e_1 \otimes e'_1) - (e_0 \otimes e'_1) \otimes (e_1 \otimes e'_0) \\ &\quad + (e_1 \otimes e'_0) \otimes (e_0 \otimes e'_1) + (e_1 \otimes e'_1) \otimes (e_0 \otimes e'_0)) \\ &= g\phi((1 \otimes f \otimes 1)[(e_0 \otimes e'_0) \otimes (e_0 \otimes e'_0) \otimes (e_1 \otimes e'_1) \\ &\quad - (e_0 \otimes e'_1) \otimes (e_0 \otimes e'_0) \otimes (e_1 \otimes e'_0) - (e_0 \otimes e'_0) \otimes (e_0 \otimes e'_1) \otimes (e_1 \otimes e'_0) \\ &\quad + (e_0 \otimes e'_0) \otimes (e_1 \otimes e'_0) \otimes (e_0 \otimes e'_1) + (e_1 \otimes e'_0) \otimes (e_0 \otimes e'_0) \otimes (e_0 \otimes e'_1) \\ &\quad + (e_0 \otimes e'_0) \otimes (e_1 \otimes e'_1) \otimes (e_0 \otimes e'_0) - (e_0 \otimes e'_1) \otimes (e_1 \otimes e'_0) \otimes (e_0 \otimes e'_0) \\ &\quad + (e_1 \otimes e'_0) \otimes (e_0 \otimes e'_1) \otimes (e_0 \otimes e'_0) + (e_1 \otimes e'_1) \otimes (e_0 \otimes e'_0) \otimes (e_0 \otimes e'_0)]) \\ &= g\phi(-e_0 \otimes e'_0 y \otimes e_1 \otimes e'_0 + e_0 \otimes e'_0 \otimes x e_0 \otimes e'_1 \\ &\quad - e_0 \otimes e'_1 \otimes x e_0 \otimes e'_0 + e_1 \otimes e'_0 y \otimes e_0 \otimes e'_0) \\ &= g((\phi_P \otimes \mu_Q \otimes 1_Q) + (1_P \otimes \mu_P \otimes \phi_Q))[-e_0 \otimes e_1 \otimes e'_0 y \otimes e'_0 - x e_0 \otimes e_1 \otimes e'_0 \otimes e'_0 \\ &\quad - e_0 \otimes e_1 x \otimes e'_0 \otimes e'_0 + e_0 \otimes x e_0 \otimes e'_0 \otimes e'_1 - e_0 \otimes x e_0 \otimes e'_1 \otimes e'_0 + e_1 \otimes e_0 \otimes e'_0 y \otimes e'_0] \\ &= g(e_1 \otimes e'_1 + e_1 \otimes e'_1) = 2y^3. \end{aligned}$$

Hence,  $[f, g](e_1 \otimes e'_1) = 3y^3 - 2y^3 = y^3$ , and we have  $[f, g] = g$ .

**Remark 4.5.** In the notation of [9, Theorem 6.6], take  $h = x^2$ ,  $n = 1$ ,  $\pi_h = x$ , and  $a_1 = xy$ . Our  $f$  is then their  $\text{ad}_{a_1}$ , and our bracket calculation agrees with theirs.

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